Source of atmospheric multiple equilibria*

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The atmosphere is a nonlinear fluid system with dissipation driven by external forcing. There exist multiple equilibria in it[1]. As early as 1958, Yeh *et al.*[2] pointed out that there exist two basic equilibria in the atmospheric circulation: winter circulation pattern and summer circulation pattern. In the late 1970s, Charnery *et al.*[3, 4] studied the multiple equilibria of barotropic atmosphere and baroclinic atmosphere with simple models, and founded the theory on the atmospheric multiple equilibria. As regards the mechanism of atmospheric multiple equilibria, based on the equations of large-scale atmospheric motion, Chou[1] pointed out that the multiple equilibria are caused by the nonlinearity. This note further studies the source of atmospheric multiple equilibria with the operator equation of the complete primitive nonlinear atmosphere.

1 Operator equation and problem description

In the spherical coordinate system (λ, θ, r, t), the complete primitive nonlinear atmosphere can be written as the following operator equation:

$$\begin{cases} \dfrac{\partial\varphi}{\partial t}+(N(\varphi)+L(\varphi))\varphi=\xi(\varphi), & (1)\\ \varphi|_{t=0}=0, & (2)\end{cases}$$

where $\varphi=(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\rho}, \tilde{T})'$ ("$'$" denotes transposition), $\varphi=\rho^*\psi=\rho^*(u^*, v^*, w^*, \sqrt{\varphi}, T^*)'$, $u^*=u/\sqrt{2}$, $v=v^*/\sqrt{2}$, $w=w^*/\sqrt{2}$, $\rho^*=\sqrt{\rho}$, $T^*=\sqrt{C_v T}$, $\varphi=gr$,

$$N(\varphi)=\begin{bmatrix} \mathscr{L} & 2\Omega\cos\theta+(u/r)\mathrm{ctg}\theta & 2\Omega\sin\theta+v/r & 0 & l_1'G \\ -2\Omega\cos\theta-(u/r)\mathrm{ctg}\theta & \mathscr{L} & v/r & 0 & l_2'G \\ -2\Omega\sin\theta-v/r & -v/r & \mathscr{L} & g/\sqrt{2\varphi} & l_3'G \\ 0 & 0 & -g/\sqrt{2\varphi} & \mathscr{L} & 0 \\ Gl_1 & Gl_2 & Gl_3 & 0 & \mathscr{L} \end{bmatrix},$$

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$$L(\varphi)=\begin{bmatrix} -(\mu_1/3)l'_1l_1-\mu_1l_4 & -(\mu_1/3)l'_1l_2 & -(\mu_1/3)l'_1l_3 & 0 & 0 \\ -(\mu_1/3)l'_2l_1 & -(\mu_1/3)l'_2l_2-\mu_1l_4 & -(\mu_1/3)l'_2l_3 & 0 & 0 \\ -(\mu_1/3)l'_3l_1 & -(\mu_1/3)l'_3l_2 & -(\mu_1/3)l'_3l_3-\mu_1l_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_2l_5 \end{bmatrix},$$

$$\xi(\varphi)=(0,\ 0,\ 0,\ 0,\ (\varepsilon+\mu_2\alpha_s T_s/C_v)/\widetilde{T})',\quad \mathscr{L}=(\Pi+\Lambda)/2,$$

$$\Pi=\frac{1}{r\sin\theta}\frac{\partial}{\partial\lambda}u+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}v\sin\theta+\frac{1}{r^2}\frac{\partial}{\partial r}r^2w,$$

$$\Lambda=\frac{u}{r\sin\theta}\frac{\partial}{\partial\lambda}+\frac{v}{r}\frac{\partial}{\partial\theta}+w\frac{\partial}{\partial r},\quad G=RT/\sqrt{2C_v},$$

$$l_1=\frac{1}{r\sin\theta}\frac{\partial}{\partial\lambda}\frac{1}{\rho^*},\quad l_2=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\frac{1}{\rho^*}\sin\theta,$$

$$l_3=\frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{\rho^*}r^2,\quad l_4=\frac{1}{\rho^*}\Delta\frac{1}{\rho^*},\quad l_5=(l_4-\alpha_s T_s/T^2)/C_v,$$

$$l'_1=(1/\rho^*)l_1\rho^*,\quad l'_2=(\sin\theta/\rho^*)l_2(\rho^*/\sin\theta),\quad l'_3=(r^2/\rho^*)l_3(\rho^*/r^2),$$

$$\Delta=\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r},$$

μ_1 is the molecular viscosity; μ_2 the turbulent thermal conductivity; $T_s=T_s(\lambda,\ \theta)$, the temperature on the surface of the earth, $\alpha_s\in L^\infty(S^2)\cap\mathbb{R}_+$; all other notations are usual meteorologically. The domain of solutions $\Omega=S^2\times(r_s,\ r_\infty)$ with $0<r_s<r_\infty<\infty$, here $r_s=r_s(\lambda,\ \theta)$ is the distance between the surface of the earth at the longitude of λ and the colatitude of θ and the geocentric, r_∞ is a certain large number.

This note studies the problem of multiple equilibria, that is, on the solutions of the boundary value problem for the equations of stationary atmospheric motion. Therefore, eq. (1) becomes

$$N(\varphi)\varphi+L(\varphi)\varphi=\xi(\varphi), \tag{3}$$

where operator $L(\varphi)$ represents dissipation, $\xi(\varphi)$ the external forcing. The problem is studied from the following aspects:

1) Omitting the nonlinear, but with dissipation and external forcing. By omitting the nonlinear is meant φ in the nonlinear operators $N(\varphi)$, $L(\varphi)$ and $\xi(\varphi)$ regarded as a known function $\overline{\varphi}$. In this way, $N(\overline{\varphi})$, $L(\overline{\varphi})$ and $\xi(\overline{\varphi})$ are independent of φ. As for the selection of $\overline{\varphi}$, it is determined by the issue in question. In principle, it is only a certain average state got by the observation and is also named after basic state[1]. This way does not lose

the physical essence of the studied problem, and is called "the permissible replacement"[5]. So eq. (3) becomes

$$N(\overline{\varphi})\varphi + L(\overline{\varphi})\varphi = \xi(\overline{\varphi}). \tag{4}$$

2) Without external forcing, but with the nonlinear and dissipation, i.e.

$$N(\varphi)\varphi + L(\varphi)\varphi = 0. \tag{5}$$

3) Without dissipation, but with the nonlinear and external forcing, namely

$$N(\varphi)\varphi = \xi(\varphi). \tag{6}$$

By the system with external forcing here is meant $\int_\Omega (\varepsilon + \mu_2 \alpha_s T_s / C_v) \mathrm{d}\Omega > 0$; that is to say, the system always obtains the energy from outside.

2 Main results

Let $H_0(\Omega)$ be the complete space with the inner product and the norm as follows:

$$(\varphi_1, \varphi_2) = \int_\Omega \varphi_1' \varphi_2 \mathrm{d}\Omega = \int_{r_s}^{r_\infty} \int_0^\pi \int_0^{2\pi} \varphi_1' \varphi_2 r^2 \sin\theta \mathrm{d}\lambda \mathrm{d}\theta \mathrm{d}r, \tag{7}$$

$$\|\varphi\|_0 = (\varphi, \varphi)^{1/2}, \tag{8}$$

$\forall \varphi = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\rho}, \tilde{T})'$. Apparently, $H_0(\Omega)$ is a Hilbert space.

Lemma 1. *$L(\varphi)$ is a positively definite self-adjoint operator, $N(\varphi)$ an anti-adjoint operator.*

Theorem 1. *There exists unique solution φ for equation* (4).

Lemma 2. $(L(\varphi)\varphi, \varphi) = (\tilde{L}\psi, \psi)$, $\forall \varphi = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\rho}, \tilde{T})'$, $\psi = (u^*, v^*, \rho^*, T^*)' \in H_0(\Omega)$. *Removing ρ^* in operator $L(\varphi)$ gives operator L.*

Lemma 3. *$\tilde{L}$ is a positively definite self-adjoint operator.*

Theorem 2. *There is unique zero solution for equation* (5).

Theorem 3. *There does not exist any solution for equation* (6).

3 Conclusion

According to the above results, the stationary solution for the equations of stationary atmospheric motion is either unique or non-existential and in any case there does not exist multi-solution if anyone of nonlinearity, dissipation and external forcing is dispensed with. This shows that the joint action of nonlinearity, dissipation and external forcing is the source of the atmospheric multiple equilibria. Therefore, we should consider the three actions of nonlinearity, dissipation and external forcing for the problems of atmospheric multiple equilibria, the catastrophe and bifurcation related to multiple equilibria, and the medium- and long-range weather. If not, the study is not suitable.

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