Existence of the atmosphere attractor*

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Abstract The global asymptotic behavior of solutions for the equations of large-scale atmospheric motion with the non-stationary external forcing is studied in the infinite-dimensional Hilbert space. Based on the properties of operators of the equations, some energy inequalities and the uniqueness theorem of solutions are obtained. On the assumption that external forces are bounded, the exsitence of the global absorbing set and the atmosphere attractor is proved, and the characteristics of the decay of effect of initial field and the adjustment to the external forcing are revealed. The physical sense of the results is discussed and some ideas about climatic numerical forecast are elucidated.

Keywords: operator equations, atmosphere attractor, global absorbing set, non-stationary external forcing, decay, adjustment.

The atmosphere is a forced dissipative nonlinear system. The essential characteristics of the atmospheric motion are formed by the basic actions such as frictional dissipation, thermal forcing, nonlinear advection, rotational force field and gravitational field. The motion obeys some physical laws and can be written as partial differential equations in the mathematical language. However, the nonlinear partial differential equations are too complicated to be solved analytically. Although we can use computers to carry out numerical experiment on it, the properties of final state of all possible initial values can by no means be made clear as time tends to infinity. But if the properties of its solutions are known directly by the equations themselves without solving the differential equations, we may understand a lot of the macroscopic properties of the atmosphere.

Under the stationary external forcing, Chou Jifan[1—4] studied the global asymptotic behavior of solutions for the system of nonlinear atmosphere, and proved that the system is bound to evolve into a state of an absorbing set in R^n, whatever the initial values might be. In the physical sense, that is the adjustment of the system to the external forcing. Chou's results were extended to the infinite-dimensional Hilbert space[5]. For the real atmospheric system, the external forces are non-stationary. A study of large-scale motion of weather and climate under the non-stationary external forcing is also of basic significance. Therefore, we extended the above results to R^n under the non-stationary external forcing[6]. Do the results hold true in infinite-dimensional Hilbert space in that case? This paper gives a discussion on the problem.

1 Basic equations

The present study addresses the equations of large-scale atmospheric motion in the spherical coordinate system (λ, θ, p, t). In order to save space, for their expressions see references [1,3—5,7] (the corresponding notations are all the same).

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The domain of solutions of the equations $\Omega = S^2 \times (p_0, P_s)$, with $0 < p_0 < P_s < \infty$. Here $p_0 > 0$ is a certain small number, and P_s the surface pressure. The boundary value conditions are given below.

On the surface of the earth $p = P_s$

$$V_\lambda = V_\theta = \omega = 0, \tag{1}$$

$$\frac{\partial T}{\partial p} = a_s(T_s - T), \tag{2}$$

where $T_s = T_s(\lambda, \theta, t)$ is the temperature on the surface of the earth (sea surface or land surface), and a_s is a positive constant related to turbulent thermal conductivity.

On the upper surface of the atmosphere $p = p_0$

$$\frac{\partial V_\lambda}{\partial p} = \frac{\partial V_\theta}{\partial p} = 0. \qquad \omega = 0, \qquad \frac{\partial T}{\partial p} = 0. \tag{3}$$

The initial value conditions are

$$(V_\lambda, V_\theta, T)\Big|_{t=0} = \left(V^{(0)}_\lambda, V^{(0)}_\theta, T^{(0)}\right). \tag{4}$$

2 Fundamental function spaces, operator equation, properties of operator and assumption

By introducing the vector function $\varphi = (V_\lambda, V_\theta, \omega, \Phi, T)'$ (here the sign $'$ denotes transposition) and the operators B, N and L, the partial differential equations of large-scale atmospheric motion can be written as

$$B\frac{\partial \varphi}{\partial t} + (N(\varphi) + L)\varphi = \xi(t), \tag{5}$$

where

$$B\varphi\Big|_{t=0} = B\varphi_0, \tag{6}$$

$$B = \mathrm{diag}(1, 1, 0, 0, R^2/C^2), \tag{7}$$

$$N(\varphi) = \begin{bmatrix} \Lambda & 2\Omega\cos\theta + \frac{\mathrm{ctg}\theta}{a}V_\lambda & 0 & \frac{1}{a\sin\theta}\frac{\partial}{\partial} & 0 \\ (-2\Omega\cos\theta + \frac{\mathrm{ctg}\theta}{a}V_\lambda) & \Lambda & 0 & \frac{1}{a}\frac{\partial}{\partial\theta} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial p} & \frac{R}{p} \\ \frac{1}{a\sin\theta}\frac{\partial}{\partial\lambda} & \frac{1}{a\sin\theta}\frac{\partial}{\partial\theta}\sin\theta & \frac{\partial}{\partial p} & 0 & 0 \\ 0 & 0 & -\frac{R}{p} & 0 & \frac{R^2}{C^2}\Lambda \end{bmatrix}, \tag{8}$$

$$L = \mathrm{diag}(L_1, L_1, 0, 0, L_2 + l_2 a_s T_s^2/T^2), \tag{9}$$

$$\xi(t) = (0,0,0,0,R^2\varepsilon(t)/C^2C_p + l_2\alpha_s T_s^2(t)/T. \quad (10)$$

where $L_i = -\partial_p l_i \partial_p - \mu_i \nabla^2$, $l_i = \nu_i(gp/R\overline{T})$, $i=1,2$. Operator $N(\varphi)$ embodies the actions of the nonlinear advection, the Coriolis force, the spherical action and the gravity, etc.; L shows the dissipation terms.

On the set formed by the whole vector function $\varphi = (V_\lambda, V_\theta, \omega, \Phi, T)'$, we define inner product and norm as follows:

$$(\varphi_1, \varphi_2) = \int_\Omega \varphi'_1 \varphi_2 \mathrm{d}\Omega = \int_{P_0}^{Ps}\int_0^{\pi}\int_0^{2\pi} \varphi'_1\varphi_2 a^2 \sin\theta \mathrm{d}\lambda \mathrm{d}\theta \mathrm{d}p, \quad (11)$$

$$\|\varphi\|_0 = (\varphi, \varphi)^{1/2}. \quad (12)$$

So we get a Hilbert space H_0.

Let B^*, L^* and $N^*(\varphi)$ be the adjoint operators of B, L and $N(\varphi)$, respectively. Then we have

Property 1. $B = B^*$, $L = L^*$, $N(\varphi) = -N^*(\varphi)$. (13)

We call B and L the self-adjoint operators, and $N(\varphi)$ the anti-adjoint operator.

Property 2. B and L are the positively definite operators,

$$(\varphi, B\varphi) \geqslant 0, \quad (14)$$

$$(\varphi, L\varphi) \geqslant 0, \quad (15)$$

$$(\varphi, N(\varphi_1)\varphi) = 0, \quad (16)$$

$\forall \varphi, \varphi_1 \in H_0(\Omega)$, the equalities in (14) and (15) are true if and only if $\|\varphi\|_0 = 0$.

(14) shows that $(\varphi, B\varphi)$ represents energy. (15) shows that the self-conjugate and positively definite properties of operator L embody the characterization that the dissipative action always dissipates energy. (16) shows that the anti-adjoint property of $N(\varphi)$ embodies the important essence that the actions of the nonlinear advection, the Coriolis force, the spherical action and the gravity do not change the total energy of the system.

Under the adiabatic condition without friction, according to (14)—(16), we know (5) has the conservation of total energy, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi, B\varphi) = 0, \quad (17)$$

namly

$$\|B_1\varphi\|_0^2 = \int_\Omega (V_\lambda^2 + V_\theta^2 + \frac{R^2}{C^2}T^2)\mathrm{d}\Omega = \text{const}, \quad (18)$$

where $B_1 = \mathrm{diag}(1,1,0,0,R/C)$.

Let $H_1(\Omega)$ be the complete space with the norm as follows:

$$\|\varphi\|_1 = (\|V_\lambda\|^2 + \|V_\theta\|^2 + \|\omega\|^2 + \|\Phi\|^2 + \|T\|^2)^{1/2}, \quad (19)$$

$\forall \varphi = (V_\lambda, V_\theta, \omega, \Phi, T)'$, where $\|V_\lambda\|$, $\|V_\theta\|$ and $\|T\|$ take $H^1(\Omega)$-norm, $\|\omega\|$ and $\|\Phi\|$ take $Q(\Omega)$-norm. Here $H^1(\Omega)$ is the standard Sobolev space. $Q(\Omega)$ is the complete space with the norm as follows:

$$\| q \| = \left(\int_{\Omega}(q^2 + (\partial q/\partial p)^2)\mathrm{d}\Omega\right)^{1/2}, \quad (q = \omega \text{ or } \Phi). \tag{20}$$

In $Q(\Omega)$, we can use the following equivalent norm:

$$\| q \| = \left(\int_{\Omega}(\partial q/\partial p)^2\mathrm{d}\Omega\right)^{1/2}, \quad (q = \omega \text{ or } \Phi), \tag{21}$$

Lemma 1. *There exist constants $K_1, K_2 > 0$ such that*

$$K_1(\| V_\lambda \|^2 + \| V_\theta \|^2 + \| T \|^2) \leqslant \| \varphi \|_1 \leqslant K_2(\| V_\lambda \|^2 + \| V_\theta \|^2 + \| T \|^2), \tag{22}$$

$\forall \varphi = (V_\lambda, V_\theta, \omega, \Phi, T)' \in H_1(\Omega)$.

So we can use the following equivalent norm in $H_1(\Omega)$:

$$\| \varphi \|_1 = (\| V_\lambda \|^2 + \| V_\theta \|^2 + \| T \|^2)^{1/2}. \tag{23}$$

In the following discussion, operator $N(\varphi)$ should be decomposed into $N^{(1)}(\varphi)$ and $N^{(2)}$:

$$N(\varphi) = N^{(1)}(\varphi) + N^{(2)} \tag{24}$$

$$N^{(1)}(\varphi) = \begin{bmatrix} \Lambda & \frac{\mathrm{ctg}\theta}{a}V_\lambda & 0 & 0 & 0 \\ -\frac{\mathrm{ctg}\theta}{a}V_\lambda & \Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{R^2}{C^2}\Lambda \end{bmatrix},$$

$$N^{(2)} = \begin{bmatrix} 0 & 2\Omega\cos\theta & 0 & \frac{1}{a\sin\theta}\frac{\partial}{\partial\lambda} & 0 \\ -2\Omega\cos\theta & 0 & 0 & \frac{1}{a}\frac{\partial}{\partial\theta} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial p} & \frac{R}{p} \\ \frac{1}{a\sin\theta}\frac{\partial}{\partial\lambda} & \frac{1}{a\sin\theta}\frac{\partial}{\partial\theta}\sin\theta & \frac{\partial}{\partial p} & 0 & 0 \\ 0 & 0 & -\frac{R}{p} & 0 & 0 \end{bmatrix}.$$

Both $N^{(1)}(\varphi)$ and $N^{(2)}$ are anti-adjoint operators.

Lemma 2. $N(\varphi_1 + \varphi_2) = N^{(1)}(\varphi_1 + \varphi_2) + N^{(2)}$ (25)

$$N^{(1)}(\alpha\varphi_1 + \beta\varphi_2) = \alpha N^{(1)}(\varphi_1) + \beta N^{(1)}(\varphi_2), \ \forall \varphi_1, \varphi_2 \in H_0(\Omega). \tag{26}$$

Lemma 3. *There exist constants $C1, C2 > 0$ such that*

$$C_1 \| \varphi \|_1^2 \leqslant (\varphi, L\varphi), \ \forall \varphi \in H_1(\Omega). \tag{27}$$

Lemma 4. *There exists a constant C such that*

$$| (N^{(1)}(\varphi)\varphi_1, \varphi) | = | (N^{(1)}(\varphi)\varphi, \varphi_1) | \leqslant C \begin{cases} \| \varphi \|_1 \ \| B_0\varphi \|_0 \| \varphi_1 \|_3, \\ \| \varphi \|_1 \| B_0\varphi \|_0 \| B_1\varphi_1 \|_3. \end{cases} \tag{28}$$

In this paper, the exteranl forcing is non-stationary. So $\xi = \xi(t)$ as a function of time. In reality, the external forces are always bounded. Therefore, we assume that the external forces are bounded in subsequent sections, namely

$$0 < \| \zeta(t) \|^2 \leqslant M < \infty, \tag{29}$$

where $\| \xi(t) \|^2 = \| R^2\varepsilon(t)/C^2C_p \|^2 + l_2\alpha_s \| T_s(t) \|^2$. $\varepsilon(t)$, $T_s(t)$ may be the quasi-periodic or the asymptotically almost periodic or the functions that can be expanded by Fourier series.

3 Energy inequalities and uniqueness of solutions

Theorem 1. *Any solution φ of the operator equations* (5) *and* (6) *satisfies*

$$\| B_1\varphi \|_0^2 + 2C_1\int_0^t \| \varphi(t) \|_1^2 dt$$
$$\leqslant \| B_1\varphi_0 \|_0^2 + 2\int_0^t (\xi(t), \varphi(t)) dt, \qquad t \in [0, T], a.e. \tag{30}$$

where C_1 is given by (27).

Furthermore, if $\xi(t) \in H_1^*(\Omega)$, where $H_1^*(\Omega)$ is the dual space of $H_1(\Omega)$, then

$$(\xi, \varphi) \leqslant \frac{1}{2C_1} \| \zeta \|_{H_1^*}^2 + \frac{C_1}{2} \| \varphi \|_1^2. \tag{31}$$

Therefore,

$$\| B_1\varphi \|_0^2 + C_1\int_0^t \| \varphi(t) \|_1^2 dt$$
$$\leqslant \| B_1\varphi_0 \|_0^2 + \frac{1}{C_2}\int_0^t \| \zeta(t) \|_{H_1^*}^2 dt, \qquad t \in [0, T], \text{a.e.} \tag{32}$$

On the other hand, using $\| B_1\varphi \|_0^2 \leqslant C_1^* \| \varphi \|_1^2$ and

$$| (\xi(t), \varphi(t)) | \leqslant \frac{1}{C_2} \| \zeta(t) \|^2 + \frac{C_1}{2} \| \varphi(t) \|_0^2,$$

we have

$$\frac{d}{dt} \| B_1\varphi \|_0^2 + \tilde{C}_1 \| B_1\varphi(t) \|_0^2 \leqslant \frac{1}{\tilde{C}_2} \| \zeta(t) \|^2. \tag{33}$$

Applying the classical Gronwall inequality, we get

Theorem 2. *Any solution φ of* (5) *and* (6) *satisfies*

$$\| B_1\varphi \|_0^2 \leqslant \left\{ \| B_1\varphi_0 \|_0^2 + \frac{1}{C_2}\int_0^t e^{\tilde{C}_1 t} \| \zeta(t) \|^2 dt \right\} e^{-\tilde{C}_1 t}, \quad (34)$$

$t \in [0, T]$, *a.e.*, $C_2 \tilde{C}_1 > 0$.

(34) has obvious physical sense. On its right-hand side, the first term shows the effect of initial value, and the second term shows the effect of the external forcing. As the time $t \to \infty$, we obtain

$$\| B_1\varphi_0 \|_0^2 e^{-\tilde{C}_1 t} \longrightarrow 0, \quad (35)$$

$$\lim_{t\to\infty} \| B_1\varphi \|_0^2 \leqslant \lim_{t\to\infty} \left\{ \| B_1\varphi_0 \|_0^2 + \frac{1}{C_2}\int_0^t e^{\tilde{C}_1 t} \| \zeta(t) \|^2 dt \right\} e^{-\tilde{C}_1 t}$$
$$= \lim_{t\to\infty}\frac{1}{C_2}\left\{ \int_0^t e^{\tilde{C}_1 t} \| \zeta(t) \|^2 dt \right\} e^{-\tilde{C}_1 t}. \quad (36)$$

They show in the general sense that the system described by (5) has the characteristic of the decay of effect of initial field[7], and that the long-range evolution of the system will be dependent on the change of external forcing.

By use of assumption (29), we have

$$\| B_1\varphi \|_0^2 \leqslant \left\{ \| B_1\varphi_0 \|_0^2 + \frac{M}{C_2}\int_0^t e^{\tilde{C}_1 t} dt \right\} e^{-\tilde{C}_1 t}$$
$$= \| B_1\varphi_0 \|_0^2 e^{-\tilde{C}_1 t} + \frac{M}{C_2\tilde{C}_1}(1 - e^{-\tilde{C}_1 t}), \qquad t \in [0, T], \text{a.e.} \quad (37)$$

Theorem 3. *There is a unique smoothing solution of the initial-boundary value problem of* (5), (6) *and* (1)—(3).

Proof. Let $\varphi_1 = (V_{1\lambda}, V_{1\theta}, \omega_1, \Phi_1, T_1)'$, $\varphi_2 = (V_{2\lambda}, V_{2\theta}, \omega_2, \Phi_2, T_2)'$ be solutions of the initial-boundary value problem of (5) and (6). Besides, let

$$\varphi_1 - \varphi_2 = \varphi = (V_\lambda, V_\theta, \omega, \Phi, T)'.$$

So by the results mentioned before, we get

$$\frac{\partial}{\partial t}B\varphi + N^{(1)}(\varphi_1)\varphi_1 - N^{(1)}(\varphi_2)\varphi_2 + N^{(2)}\varphi + L^*\varphi = 0, \quad (38)$$

$$\varphi(\lambda, \theta, p; 0) = 0, \quad (39)$$

$$(V_\lambda, V_\theta, \omega) = 0, \partial T/\partial p = -\alpha_s T, \qquad \text{on } p = P_s, \quad (40)$$

$$(\partial V_\lambda/\partial p, \partial V_\theta/\partial p, \omega, \partial T/\partial p) = 0, \qquad \text{on } p = p_0 \quad (41)$$

where $L^* = \mathrm{diag}(L_1, L_1, 0, 0, L_2)$.

By Lemma 2, we have

$$\frac{\partial}{\partial t}B\varphi + N^{(1)}(\varphi + \varphi_2)\varphi + N^{(1)}(\varphi)\varphi_2 + N^{(2)}\varphi + L^*\varphi = 0. \quad (42)$$

Therefore,

$$\frac{d}{dt}\| B_1\varphi \|_0^2 + 2(N^{(1)}(\varphi + \varphi_2)\varphi, \varphi) + 2(N^{(1)}(\varphi)\varphi_2, \varphi) + 2(N^{(2)}\varphi, \varphi) + 2(L^*\varphi, \varphi) = 0. \tag{43}$$

So

$$\frac{d}{dt}\| B_1\varphi \|_0^2 + 2C_1 \| \varphi \|_1^2 \leqslant 2 \mid N^{(1)}(\varphi)\varphi_2, \varphi) \mid$$

$$\leqslant 2C \| \varphi \|_1 \| B_1\varphi \|_0 \| B_1\varphi_2 \|_3 \leqslant C_1 \| B_1\varphi \|_1^2 + \frac{C^2}{C_1} \| B_1\varphi \|_0^2 \| B_1\varphi_2 \|_3^2, \tag{44}$$

which implies

$$\frac{d}{dt}\| B_1\varphi \|_0^2 \leqslant \tilde{C} \| B_1\varphi_2 \|_3^2 \| B_1\varphi \|_0^2. \tag{45}$$

Integrating (45) and using (39), we have immediately $\| B_1\varphi \|_0^2 \equiv 0$, i.e. $\varphi_1 \equiv \varphi_2$. Q.E.D.

4 Existence of the atmosphere attractor

Theorem 4. *Under assumption* (29), *let*

$$B_K = \{\varphi = (V_\lambda, V_\theta, \omega, \Phi, T)' \in H_0(\Omega) \mid \| B_1\varphi \|_0^2 \leqslant K\}, \tag{46}$$

$$\tau = \frac{1}{\tilde{C}_1}\ln\frac{\| B_1\varphi_0 \|_0^2 - M_1}{K - M_1}, \tag{47}$$

$$K > M_1 \tag{48}$$

$$M_1 = M/C_2\tilde{C}_1, \tag{49}$$

where C_2 is given by (31) *and $\tilde{C}_1$ is given by* (33). *The solutions of* (5) *and* (6) *statisfy*: (i) *if $B_1\varphi_0 \in B_K$, then for $\forall\, t \geqslant 0$, $B_1\varphi(t) \in B_K$*; (ii) *if $B_1\varphi_0 \notin B_K$, then for $\forall\, t \geqslant \tau$, $B_1\varphi(t) \in B_K$*.

Proof. Under assumption (29), a solution of (5) and (6) satisfies

$$\| B_1\varphi \|_0^2 \leqslant \| B_1\varphi_0 \|_0^2 e^{-\tilde{C}_1 t} + M_1(1 - e^{-\tilde{C}_1 t}). \tag{50}$$

Because $0 < e^{-\tilde{C}_1 t} \leqslant 1$, $\forall\, t \geqslant 0$. We obtain $(K - M_1)e^{-\tilde{C}_1 t} \leqslant K - M_1$, which implies

$$Ke^{-\tilde{C}_1 t} + M_1(1 - e^{-\tilde{C}_1 t}) \leqslant K - M_1. \tag{51}$$

If $B_1\varphi_0 \in B_K$, then $\| B_1\varphi_0 \|_0^2 \leqslant K$. From (50) and (51), we get

$$\| B_1\varphi \|_0^2 \leqslant Ke^{-\tilde{C}_1 t} + M_1(1 - e^{-\tilde{C}_1 t}) \leqslant K.$$

Hence $\forall\, t \geqslant 0$, we have $B_1\varphi(t) \in B_K$.

If $B_1\varphi_0 \notin B_K$, then $\| B_1\varphi_0 \|_0^2 > K$. As $\forall\, t \geqslant \tau$, from (50) we have

$$\| B_1\varphi \|_0^2 \leqslant \frac{K - M_1}{\| B_1\varphi_0 \|_0^2 - M_1} \| B_1\varphi_0 \|_0^2 + M_1\left(1 - \frac{K - M_1}{\| B_1\varphi_0 \|_0^2 - M_1}\right) = K.$$

Therefore $B_1\varphi(t) \in B_K$. Q.E.D.

Theorem 4 shows that there exists a bounded sphere B_K in H_0 such that the solution $\varphi(t)$ of (5) which takes any point out of B_K as initial value is bound to run into the B_K and remain forever and the solution of (5) which takes any point inside of B_K cannot run away from B_K; that is to say, $B_1\varphi(t) \in B_K$ is sure for any possible initial value $B_1\varphi_0$ as $t \geqslant \tau$. Therefore, we call B_K the global absorbing set. Points out of B_K that have nothing to do with the asymptotic state (i.e. the long-range behavior) of the system will only depend on the bounded sphere B_K. The lower bound of K depends on the maximum M of the external forcing. What is the final set of B_K? Can it tend to an invariant point set or not? The subsequent discussion will focus on these questions.

According to Theorem 3, $B_1\varphi(t)$ is uniquely fixed by the initial value $B_1\varphi_0$. So (5) and (6) define the mapping (or the solution operator) $S(t)$: $H_0 \to H_0$ such that $S(t)B_1\varphi_0 = B_1\varphi(t)$. We define

$$S(t)R = \{S(t)B_1\varphi_0 \mid \forall B_1\varphi_0 \in R \subset H_0\}. \tag{52}$$

Therefore, Theorem 4 can be rewritten as follows.

Theorem 4′. *B_K is an absorbing set, i.e. for $\forall R \subset H_0$, R is a bounded set, and there exists a $\tau(R)$ such that*

$$S(t)R \subset B_K, \qquad \forall t \geqslant \tau(R). \tag{53}$$

Lemma 5 (see ref.[8]). *Suppose that H is a metric space, and that $s(t)$ is a continuous semigroup and uniform compact for large t. Moreover, let U be an open set and $\mathscr{A}$ be a bounded set of U so that $\mathscr{A}$ is an absorbing set in U. Then*

$$A = \bigcap_{S \geqslant 0} \bigcup_{t \geqslant S} S(t)\mathscr{A} \tag{54}$$

is a compact attractor which attracts any bounded set in U and is the largest attractor of U.

We define

$$A = \bigcap_{S \geqslant 0} \overline{\bigcup_{t \geqslant S} S(t)B_K}. \tag{55}$$

Then we have

Theorem 5. *A satisfies*

(i) *A is a bounded set in H_0;*

(ii) *A is a functional invariant set of $S(t)$, i.e.*

$$S(t)A = A, \qquad \forall t \geqslant 0; \tag{56}$$

(iii) *there exists an open neighbourhood U of A such that for any $\forall B_1\varphi_0 \in U$, there is $S(t)B_1\varphi_0 \to 0$ as $t \to \infty$, namely*

$$\mathrm{dist}(S(t)B_1\varphi_0, A) \to 0, \quad as\ t \to \infty, \tag{57}$$

where $\mathrm{dist}(x, A) = \inf_{y \in A} \mathrm{d}(x, y)$, $\mathrm{d}(x, y)$ *is the distance between* x *and* y *in* H_0;

(iv) A *uniformly attracts set* B_K;

$$\mathrm{d}(S(t)B_K, B_K) \to 0, \quad \text{as } t \to \infty, \tag{58}$$

where $\mathrm{d}(A_0, A_1) = \sup_{x \in A_0} \inf_{y \in A_1} \mathrm{d}(x, y)$;

(v) A *is a global attractor of* $S(t)$.

The above conclusions show that there exists the global attractor A such that the atmosphere system governed by (5) and (6) is closer and closer to A. The global attractor A shows the final state of the system. We call it the atmosphere attractor. Besides, as open physical systems, the long-range weather system and the climate system have the nonlinear adjusted property to the external forcing, because the effect of initial value decays and the long-range behavior of the system only depends on the external forcing, as is pointed out by (36) and Theorem 4.

5 Summary and discussion

In this paper, the global asymptotic behavior of the equations of the large-scale atmosphere motion with the non-stationary external forcing is studied in the infinite-dimensional Hilbert space. On the assumption that the external forces satisfy (29), we prove the existence of the global absorbing set B_K and the atmosphere attractor A; i.e. the atmosphere system described by (5) and (6) will run into the global absorbing set B_K as the time τ is greater than a certain critical time and tends towards the global attractor A as the time increases. That is to say, the long-range evolution of the atmosphere and the climate are in a state of attractor. Moreover, the system has the characteristic of the decay of the effect of initial value. According to the above conclusions, we think that the following points are worthy of our attention.

(i) Since the Hausdorff dimensions of attractor are finite, the partial differential equations of atmosphere can be described exactly by a set of finite-dimensional ordinary differential equations. In this way, the problem becomes simple. However, the way of how to estimate the dimensions of system is very important and critical. We need to carry out strict theoretical estimation of the dimensions of attractor because there are some serious problems in the methods of estimating the dimensions of the attractor by use of experimental data[6].

(2) We can prove that the phase volume of the above attractor is zero in R^n. This shows that the initial values determined by observational data are not sure in the state of attractor and not in accordance with the actual situation of the atmosphere and do not match the model equations because of the error of observation caused by the instrumental precision and the error of interpolation caused by the lack of observations. That is why we should process various initial values so as to make them match the model in numerical forecast.

(3) When designing the numerical model of climatic forecast, we should utilize the property of the decay of the effect of initial field so as to simplify the problem. Meanwhile, we should fully consider and understand the change in the external forcing. Undoubtedly, this is very important

for the climatic forecast.

(4) When designing the algorithm of climatic model or simplifying the atmosphere equations, we should not change the properties of operators of the atmosphere equations so as to keep the physical essence of the system.

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