The property of solutions for the equations of large-scale atmosphere with the non-stationary external forcings*

LI Jianping (李建平) and CHOU Jifan (丑纪范)

(Department of Atmospheric Sciences, Lanzhou University, Lanzhou 730000, China)

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The long-term behavior of the atmospheric evolution, which cannot be answered and solved by the numerical experiments, must be understood before we design the numerical forecast models of the long-range weather and climate. It is necessary to carry out some studies of basic theory. Based on the stationary external forcings, Chou[1—3] studied the adjustment of the nonlinear atmospheric system tending to forcings in R^n. Then these results were extended to the infinite dimensional Hilbert space[4]. For the real atmospheric system, the external forcings are non-stationary. In such a case, are the results true or not? This note will deal with the main problem.

1 Problem, assumption and notations

The partial differential equations of large-scale atmosphere can be turned into the following operator equation:

$$\begin{cases} B\dfrac{\partial\psi}{\partial t}+(N+L)\psi=\xi, & (1) \\ B\psi=B\psi_0, \text{ when } t=t_0, & (2) \end{cases}$$

where B and L are the positively definite self-adjoint operators, N the anti-adjoint operator, both the non-homogeneous terms and the boundary conditions of the partial differential equations are included in ξ (for terms of B, N and L see references [1—3, 5]).

In R^n, B and L are the $(n\times n)$ positively definite symmetric matrix, N the $(n\times n)$ antisymmetric matrix. If we study the stationary or non-stationary external forcings, ξ is the n-dimensional constant vector or varied vector ξ_t respectively. Our problem is the asymptotic property of atmospheric motion with the non-stationary external forcings as time $t\to\infty$ in R^n, namely, the destination of the integral curve determined by the following equation as time $t\to\infty$

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$$B\frac{\partial\psi}{\partial t}+(N+L)\psi=\xi_t. \tag{3}$$

We need an assumption and some notations before discussion. In reality, the external forcings should be bounded, namely

$$0<\|\xi_t\|^2\leqslant M<\infty, \tag{4}$$

where $\|\cdot\|$ represents the norm.

For the sake of convenience, let $M_0=M(t)|_{t=t_0}$ represent one point of phase space. Then the solution to eq. (3) in which the initial value equals M_0 is $M(t)=\psi(M_0,\ t)$, the following point set

$$\gamma=\{\psi(M_0,\ t):t_0\leqslant t<\infty\}_{\text{for a certain initial value } M_0} \tag{5}$$

represents paths through M_0, and the following set of points

$$V(t^*)=(\psi(M_0,\ t):M_0\in V_0\}_{t=t^*} \tag{6}$$

may be regarded as a mapping. It maps V_0 to V, where V_0 and $V\subset R^n$.

2 Main theorems

Lemma 1. *If the ellipsoid E:* $\sum_{i=1}^{n}(\psi_i-a_i)^2/r_i=1$, *where* r_i *and* a_i $(i=1,\ 2,\cdots,\ n)$ *are constants,* $r_i\in R^+$, $a_i\in R$, *then*

i) *for any point* ψ *out of E, we have* $\sum_{i=1}^{n}(\psi_i-a_i)^2/r_i>1$;

ii) *for any point* ψ *inside of E, we have* $\sum_{i=1}^{n}(\psi_i-a_i)^2/r_i<1$.

Lemma 2. *If the ellipsoid E(A, r):* $\sum_{i=1}^{n}(\psi_i-a_i)^2/r_i=1$, *and* $A\in R_1^n$, $r\in R_2^n$, $r_i\in R^+$ $(i=1,\ 2,\ \cdots,\ n)$, *where* $A=(a_1,\ a_2,\cdots,\ a_n)$, $r=(r_1,\ r_2,\cdots,\ r_n)$, R_1^n *and* R_2^n *are the bounded closed set, then*

i) $E=\bigcup_{R_1^n}E_{(A,\ r)}$ *is a bounded set, where* $\bigcup_{R_1^n}E_{(A,\ r)}$ *represents the union of all ellipsoids that take A as the center of ellipsoid and r as the axis, A is any point in* R_1^n;

ii) *for any point* ψ *out of E,* $\sum_{i=1}^{n}(\psi_i-a_i)^2/r_i>1$.

Theorem. *Under the assumption of eq.* (4), *the solution to eq.* (3) *is satisfied, there exists a bounded closed set* V_0 *such that*

i) *if* $M_0\in V_0$, *then* $\gamma\in V_0$;

ii) $\forall M_0\notin V_0$, *there exists a* $\tau>0$, *the point set* $V\in V_0$, *where* $V_\tau=\{\psi(M_0,\ t):\ \tau\leqslant t<\infty\}$.

3 Proof of theorems

Dotting both sides of eq. (3) with ψ and using the relation $(\psi, N\psi)=0$, we have

$$\frac{\mathrm{d}(\psi, B\psi)}{\mathrm{d}t}=2[(\psi, \xi_t)-(\psi, L\psi)]. \tag{7}$$

Because L is the $(n\times n)$ positively definite symmetric matrix, there are n characterstic values $\lambda_i>0$ $(i=1, 2,\cdots, n)$. Let e_i $(i=1, 2,\cdots, n)$ be the corresponding normalized characteristic vectors. They form a set of the normalized orthogonal basis of the n-dimensional space. So let

$$\xi_t=\xi_1(t)e_1+\xi_2(t)e_2+\cdots+\xi_n(t)e_n, \tag{8}$$

$$a_t=a_1(t)e_1+a_2(t)e_2+\cdots+a_n(t)e_n, \tag{9}$$

where

$$a_i(t)=\xi_i(t)/(2\lambda_i), \tag{10}$$

then

$$2La_t=\xi_t. \tag{11}$$

Let

$$\psi=\psi_1e_1+\psi_2e_2+\cdots+\psi_ne_n. \tag{12}$$

Because

$$Le_i=\lambda_i e_i, \tag{13}$$

$$(\psi, \xi_t)-(\psi, L\psi)=\sum_{i=1}^{n}\lambda_i\{a_i^2(t)-[a_i(t)-\psi_i]^2\}. \tag{14}$$

Consider the point set E_1: $(\psi, \xi_t)-(\psi, L\psi)=0$, namely, the point set satisfies the following equation:

$$\sum_{i=1}^{n}\left\{[\psi_i-a_i(t)]^2\Big/\left[\sum_{j=1}^{n}\lambda_j a_j^2(t)/\lambda_i\right]\right\}=1. \tag{15}$$

According to eqs. (4) and (10), we have $0<\|a_t\|^2\leqslant M_1<\infty$, $a_t\in R_1^n$, where R_1^n is a bounded closed set. Meanwhile, λ_i $(i=1, 2,\cdots, n)$ belong to R^+ and are bounded, so $0<\|r_t\|^2\leqslant M_2<\infty$, where $r_t=(r_1(t), r_2(t),\cdots, r_n(t))$, $r_i(t)=\sum_{j=1}^{n}\lambda_j a_j^2(t)/\lambda_i$, $r_i(t)\in R^+$. Then we have $r_t\subset R_2^n$, where R_2^n is a bounded closed set. Let $E=\bigcup_{R_1^n}E_{(a_t, r_t)}$ represent the union of all ellipsoids that satisfy eq. (15) and take a_t as the center of ellipsoid and r_t as the axis, where a_t is any point in R_1^n. As a result of Lemma 2, E is a bounded closed set; besides, $E\supset E_1$. Then according to Lemma 2, for any point out of E, we have

$$\frac{\mathrm{d}(\psi, B\psi)}{\mathrm{d}t}<0. \tag{16}$$

Let a bounded closed set $V_0\supset E$. Then ψ satisfies eq. (16), where ψ is any point out of V_0. Then there exists a certain time τ, paths through any point out of V_0 will run into V_0 as time $t>\tau$, and paths through any point inside of V_0 cannot run away from V_0. It is evident that there exists a bounded closed set V_0 that satisfies $V_0\supset E$. Then the theorem has been proved.

4 Discussion

According to the above discussion, for the non-stationary external forcings with condition (4), the atmospheric system will run into the attractive set of points V_0 while the time t is greater than a certain critical time τ. Points out of V_0 have nothing to do with the asymptotic behavior for the time $t \to \infty$ and have only transient sense. The long-term behavior of the system will depend on the bounded closed set V_0. The result shows that the long-range weather or climate is in a state of attractor. Because the contracted evolution of dissipative system from the high-dimensional phase space to the low-dimensional attractor is a process that merges degrees of freedom, the effective degrees of freedom that determine the long-term behavior of the system are eventually reduced to limited degrees. We can make an estimate of dimension of the large-scale atmospheric attractor with the non-stationary external forcings in theory using the property of long-term weather. It is a great help to build a new theory and computational methods of long-range numerical forecast. This will be reported in other papers.

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