# Asymptotic behavior of the solutions of the atmospheric equations with topographic effect*

LI Jianping (李建平)

(State Key Laboratory of Numerical Modelling for Atmospheric Sciences and Geophysical Fluid Dynamics, Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing 100029, China)

and CHOU Jifan (丑纪范)

(Department of Atmospheric Sciences, Lanzhou University, Lanzhou 730000, China)

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**Abstract** Using the properties of integral with parameters, the characteristics of the operators of the full atmospheric operator equation are studied in the basic function spaces. Based on them, the asymptotic behavior of the solutions of the atmospheric equations is discussed and the existence of the global attractor of the atmospheric equations is proved. Therefore, the known results of the qualitative theory of the atmospheric equations without topographic effect are extended to the case with topographic effect.

**Keywords: topographic effect, operator equation, global attractor, asymptotic behavior, integral with parameters.**

The systematic studies on theoretical foundation in the qualitative global analysis theory of the dynamic equations of the atmospheric motion[1—15] have been carried out since the early 1980s, and some results have been obtained from the studies on the asymptotic behavior of the solutions of the forced dissipative nonlinear atmosphere system such as the existence of atmosphere attractor and the nonlinear adjustment of the atmospheric system to external forcing. Moreover, the effects of forcing, dissipation and nonlinearity on the asymptotic behavior of the solutions of the atmospheric equations and the source of atmospheric multiple equilibria have been revealed[8,10,14—15]. Some possibilities of applications were investigated based on these theoretical results[9,16—21], and proved that the qualitative theory has great prospect in the context of studies on climate. The topographic effect, however, has not been studied in the known qualitative theory because of the difficulties in the study involving the dynamical effect of topography. The results of weather diagnosis and numerical simulations suggest that there are significant effects of topographic dynamical forcing on the atmospheric motion[18,22]. Therefore, it is important to investigate the long-time behavior of the atmospheric motion with topographic effect to answer the question of whether the results in the known qualitative theory are true or not when the dynamical effect of topography is taken into account. In this paper, with the properties of integral with parameters, the asymptotic behavior of the solutions of the atmospheric equations with topographic effect is discussed.

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## 1 Problem description

In the spherical coordinate $(\theta, \lambda, r)$ ($\theta, \lambda$ and $r$ are colatitude, geocentric distance and longitude, respecitvely), introducing the vector function $\varphi = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\rho}, \tilde{T})'$, where the symbol $'$ denotes transposition, $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\rho}, \tilde{T})' = \rho^*(u^*, v^*, w^*, \phi^*, T^*)'$, $u^* = u/\sqrt{2}$, $v^* = v/\sqrt{2}$, $w^* = w/\sqrt{2}$, $\phi^* = \sqrt{\phi}$, $\rho^* = \sqrt{\rho}$, $T^* = \sqrt{C_V}\hat{T} = \sqrt{C_V T}$, where $u, v$ and $w$ are zonal, meridional and vertical winds respectively, $\rho$ and $T$ density of dry air and air temperature respectively, $C_V$ isosteric specific heat, the full equations of dry atmosphere can be written as the following equivalent operator equation:

$$\frac{\partial \varphi}{\partial t} + (N(\varphi) + L(\varphi))\varphi = \xi(\varphi), \tag{1}$$

$$\varphi\,|_{t=0} = \varphi_0. \tag{2}$$

where

$$\mathrm{N}(\varphi) = \begin{bmatrix} \mathscr{R} & f + u\mathrm{ctg}\theta/r & \tilde{f} + u/r & 0 & l'_1 G \\ -(f + u\mathrm{ctg}\theta/r) & \mathscr{R} & v/r & 0 & l'_2 G \\ -(\tilde{f} + u/r) & -v/r & \mathscr{R} & g/\sqrt{2}\phi^* & l'_3 G \\ 0 & 0 & -g/\sqrt{2}\phi^* & \mathscr{R} & 0 \\ Gl_1 & Gl_2 & Gl_3 & 0 & \mathscr{R} \end{bmatrix}, \tag{3}$$

$$\mathrm{L}(\varphi) = \begin{bmatrix} -\hat{\mu}l'_1 l_1 - \mu l_4 - l & -\hat{\mu}l'_1 l_2 & -\hat{\mu}l'_1 l_3 & 0 & 0 \\ -\hat{\mu}l'_2 l_1 & -\hat{\mu}l'_2 l_2 - \mu l_4 - l & -\hat{\mu}l'_2 l_3 & 0 & 0 \\ -\hat{\mu}l'_3 l_1 & -\hat{\mu}l'_3 l_2 & -\hat{\mu}l'_3 l_3 - \mu l_4 - l & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -l_5 - l'_5 \end{bmatrix}, \tag{4}$$

$$\xi(\varphi) = (0,0,0,0,(\varepsilon + \hat{C}\alpha_S T_S)/2\tilde{T})', \tag{5}$$

$$\mathscr{R} = \left(\prod + \Lambda\right)/2,\ \prod = \frac{1}{r\sin\theta}\frac{\partial}{\partial\lambda}u + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}v\sin\theta + \frac{1}{r^2}\frac{\partial}{\partial r}r^2 w,$$

$$\Lambda = \frac{u}{r\sin\theta}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial r},\ f = 2\Omega\cos\theta,\ \tilde{f} = 2\Omega\sin\theta,$$

$$G = R\tilde{T}/\sqrt{2}C_V,\ \phi = gr,\ \hat{\mu} = \mu/3,$$

$$l = \frac{1}{\rho^* r\sin\theta}\frac{\partial}{\partial\lambda}\frac{k_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda}\frac{1}{\rho^*} + \frac{1}{\rho^* r\sin\theta}\frac{\partial}{\partial\theta}\frac{k_\theta\sin\theta}{r}\frac{\partial}{\partial\theta}\frac{1}{\rho^*} + \frac{1}{\rho^* r^2}\frac{\partial}{\partial r}k_r r^2\frac{\partial}{\partial r}\frac{1}{\rho^*},$$

$$l_1 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\lambda}\frac{1}{\rho^*},\ l'_1 = \frac{1}{\rho^* r\sin\theta}\frac{\partial}{\partial\lambda},\ l_2 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\rho^*},\ l'_2 = \frac{1}{\rho^* r}\frac{\partial}{\partial\theta},$$

$$l_3 = \frac{1}{r^2}\frac{\partial}{\partial r}\frac{r^2}{\rho^*},\ l'_3 = \frac{1}{\rho^*}\frac{\partial}{\partial r},\ l_4 = \frac{1}{\rho^*}\Delta\frac{1}{\rho^*},$$

$$l_5 = \frac{1}{\rho^* r\sin\theta}\frac{\partial}{\partial\lambda}\frac{K_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda}\frac{1}{\rho^*} + \frac{1}{\rho^* r\sin\theta}\frac{\partial}{\partial\theta}\frac{K_\theta\sin\theta}{r}\frac{\partial}{\partial\theta}\frac{1}{\rho^*} + \frac{1}{\rho^* r^2}\frac{\partial}{\partial r}K_r r^2\frac{\partial}{\partial r}\frac{1}{\rho^*} - C_K\alpha_S T_S/2\tilde{T}^2,$$

$$l'_5 = \frac{1}{\tilde{T}^2}\left[\frac{\kappa_\lambda}{r^2\sin^2\theta}\left(\frac{\partial\hat{T}}{\partial\lambda}\right)^2 + \frac{\kappa_\theta}{r^2}\left(\frac{\partial\hat{T}}{\partial r}\right)^2 + \kappa_r\left(\frac{\partial\hat{T}}{\partial r}\right)^2\right] + \frac{\kappa_\theta}{C_V\rho^* r^2\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}\frac{1}{\rho^*}$$

$$+ \frac{\kappa_\lambda}{C_V\rho^* r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\frac{1}{\rho^*} + \frac{\kappa_r}{C_V\rho^* r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\frac{1}{\rho^*} - C_\kappa\alpha_S T_S/2\tilde{T}^2,$$

$$\hat{C} = C_K + C_\kappa,\ C_K = C_V K_{r_S} r_S^2 C_r^{-1},\ C_\kappa = \kappa_{r_S} r_S^2 C_r^{-1},\ C_r = \int_{r_S}^{r_\infty} r^2 dr, \qquad (6)$$

where $\varepsilon$ represents diabatic heating, $\upsilon = \mu/\rho$ molecular kinematic viscosity coefficient, $\mu$ molecular dynamic viscosity coefficient, $\kappa_\lambda$, $\kappa_\theta$ and $\kappa_r$ denote molecular thermal diffusion coefficients in horizontal and vertical directions respectively, $k_\lambda$, $k_\theta$ and $k_r$ represent turbulent viscosity coefficients in horizontal and vertical directions respectively; $K_\lambda$, $K_\theta$, $K_r$ are turbulent conductivity in horizontal and vertical directions, respectively. The domain of solutions is $\Omega = S^2 \times (r_S, r_\infty)$, in which $r_S = r_S(\theta, \lambda)$ is the distance between the surface of the earth and the geocentric at the colatitude of $\theta$ and the longitude $\lambda$, $r_\infty$ is arbitrary large number, and $0 < r_S < r_\infty < \infty$. The boundary value conditions (BVCs) are as follows:

(i) Lower BVCs: on the earth's surface $r = r_S$, while regarding the topographic effect, one has

$$w = \frac{u}{r_S\sin\theta}\frac{\partial r_S}{\partial\lambda} + \frac{v}{r_S}\frac{\partial r_S}{\partial\theta}, \qquad (7)$$

$$(\partial u/\partial r,\ \partial v/\partial r) = (C_{DK}u,\ C_{DK}v), \qquad (8)$$

$$\partial T/\partial r = \alpha_S(T - T_S), \tag{9}$$

where $C_{DK} = C_{DK}(\theta, \lambda)$ is a coefficient related to drag coefficient (depends on surface roughness and stability, etc.) and turbulent eddy coefficient, $T_S = T_S(\theta, \lambda)$, the temperature on the surface of the earth (the sea and land surfaces), $\alpha_S = \alpha_S(\theta, \lambda)$ a parameter related to turbulent thermal conductivity.

(ii) Upper BVCs: on the top of atmosphere $r = r_\infty$, one has

$$(\rho u^2, \rho v^2, \rho w^2, \rho\phi, \rho T) = 0, \tag{10}$$

$$(\partial u/\partial r,\ \partial v/\partial r,\ w = 0,\ \partial T/\partial r) = 0. \tag{11}$$

As topographic effect is considered, the surface of the earth is not the isoradius sphere which is considered in the known qualitative theory but the function of $\theta$ and $\lambda$, i.e. $r_S = r_S(\theta, \lambda)$. Therefore, the basic functional spaces used need to be redefined and also the corresponding operators need to be restudied. The aim of this work is to study the asymptotic properties of eq. (1) under the initial- and boundary-value conditions (2) and (7)—(11) on the domain $\Omega$.

## 2 Basic spaces and lemmas

Let $H_0(\Omega)$ be the complete space with the interior product and the norm as follows:

$$(\varphi_1, \varphi_2) = \int_\Omega \varphi'_1 \varphi_2 \mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi}\int_{r_s}^{r_\infty} \varphi'_1 \varphi_2 r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\lambda, \tag{12}$$

$$\| \varphi \|_0 = (\varphi, \varphi)^{1/2}. \tag{13}$$

Let $H_1(\Omega)$ be the complete space with the following norm

$$\| \Psi \|_1 = (\| u^* \|^2_{H^1(\Omega)} + \| v^* \|^2_{H^1(\Omega)} + \| w^* \|^2_{H^1(\Omega)} + \| \rho^* \|^2_{H^1(\Omega)} + \| T^* \|^2_{H^1(\Omega)}), \tag{14}$$

where $\Psi = (u^*, v^*, w^*, \rho^*, T^*)'$, $\| \cdot \|_{H^1(\Omega)}$ takes $H^1(\Omega)$-norm, and $H^1(\Omega)$ is the standard Sobolev space.

**Lemma 1.** *In $H_0(\Omega)$, one can use the following equivalent norm*

$$\| \varphi \|_0 = (\| \tilde{u} \|^2 + \| \tilde{v} \|^2 + \| \tilde{w} \|^2 + \| \rho^* \|^2 + \| \tilde{T} \|^2)^{1/2}. \tag{15}$$

Let $\mathrm{N}^*(\varphi)$ and $\mathrm{L}^*(\varphi)$ be the adjoint operators of $\mathrm{N}(\varphi)$ and $\mathrm{L}(\varphi)$ respectively. Then we have

**Lemma 2.** $\mathrm{N}(\varphi) = -\mathrm{N}^*(\varphi)$, $\mathrm{L}(\varphi) = \mathrm{L}^*(\varphi)$, $\forall \varphi \in H_0(\Omega)$.

We call $\mathrm{N}(\varphi)$ the anti-adjoint operator and $\mathrm{L}(\varphi)$ the self-adjoint operator.

*Proof*. Since

$$(N(\varphi)\varphi,\varphi^*) = \int_\Omega \Big\{ [\mathscr{R}\tilde{u} + (f + uctg\theta/r)\tilde{v} + (\tilde{f} + u/r)\tilde{w} + l'_1 G\tilde{T}]\tilde{u}^*$$

$$+ [-(f + uctg\theta/r)\tilde{u} + \mathscr{R}\tilde{v} + v\tilde{w}/r + l'_2 G\tilde{T}]\tilde{v}^*$$

$$+ [-(\tilde{f} + u/r)\tilde{u} - v\tilde{v}/r + \mathscr{R}\tilde{w} + g\tilde{\rho}/(\sqrt{2}\phi^*) + l'_3 G\tilde{T}]\tilde{w}^*$$

$$+ [-gw/(\sqrt{2}\phi^*) + \mathscr{R}\tilde{\rho}]\rho^* + [Gl_1\tilde{u} + Gl_2\tilde{v} + Gl_3\tilde{w} + \mathscr{R}\tilde{T}]\tilde{T}^* \Big\} d\Omega, \quad (16)$$

it is only necessary to prove the following three formulas in order to prove the first part of Lemma 2,

$$\int_\Omega (\mathscr{R}\tilde{F})\tilde{F}^* d\Omega = -\int_\Omega \tilde{F}(\mathscr{R}\tilde{F}^*) d\Omega, \quad (17)$$

where $F = u, v, w, \rho$ or $T$,

$$\int_\Omega [(l'G_1\tilde{T})\tilde{u}^* + (l'_2 G\tilde{T})\tilde{v}^* + (l'_3 G\tilde{T})\tilde{w}^*] d\Omega$$

$$= -\int \tilde{T}[Gl_1\tilde{u}^* + Gl_2\tilde{v}^* + Gl_3\tilde{w}^*] d\Omega, \quad (18)$$

$$\int_\Omega [Gl_1\tilde{u} + Gl_2\tilde{v} + Gl_3\tilde{w}]\tilde{T}^* d\Omega = -\int_\Omega [\tilde{u}(l'_1 G\tilde{T}^*) + \tilde{v}(l'_2 G\tilde{T}^*) + \tilde{w}(l'_3 G\tilde{T}^*)] d\Omega. \quad (19)$$

For formula (17), one only has to prove the case of $F = u$, namely

$$\int_\Omega (\mathscr{R}\tilde{u})\tilde{u}^* d\Omega = \int_\Omega [2^{-1}(\prod + \Lambda)\tilde{u}]\tilde{u}^* d\Omega$$

$$= -\int_\Omega \tilde{u}[2^{-1}(\prod + \Lambda)\tilde{u}^*] d\Omega = -\int_\Omega \tilde{u}(\mathscr{R}\tilde{u}^*) d\Omega. \quad (20)$$

The proofs of the other cases of $F$ are similar to it.

Since

$$\int_\Omega \frac{\tilde{u}^*}{r\sin\theta}\frac{\partial u\tilde{u}}{\partial\lambda} d\Omega = \int_0^{2\pi}\int_0^{\pi}\int_{r_s}^{r_\infty} \frac{\tilde{u}^*}{r\sin\theta}\frac{\partial u\tilde{u}}{\partial\lambda} r^2\sin\theta \, dr d\theta d\lambda$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_{r_s}^{r_\infty} \Big[\frac{\partial(r\tilde{u}^* u\tilde{u})}{\partial\lambda} - \frac{u\tilde{u}}{r\sin\theta}\frac{\partial\tilde{u}^* u}{\partial\lambda} r^2\sin\theta\Big] dr d\theta d\lambda$$

$$= \int_0^{2\pi}\int_0^{\pi}\Big[\frac{\partial}{\partial\lambda}\Big(\int_{r_s}^{r_\infty} r\tilde{u}^* u\tilde{u}\,\mathrm{d}r\Big) + r\tilde{u}^* u\tilde{u}\frac{\partial r}{\partial\lambda}\,|_{r=r_s}\Big]\mathrm{d}\theta\mathrm{d}\lambda - \int_\Omega \frac{\tilde{u}u}{r\sin\theta}\frac{\partial\tilde{u}^*}{\partial\lambda}\mathrm{d}\Omega$$

$$= \int_0^{2\pi}\int_0^{\pi}(r\tilde{u}^* u\tilde{u}\partial r/\partial\lambda)\,|_{r=r_s}\mathrm{d}\theta\mathrm{d}\lambda - \int_\Omega \tilde{u}ur^{-1}\sin^{-1}\theta\partial\tilde{u}^*/\partial\lambda\mathrm{d}\Omega, \tag{21}$$

$$\int_\Omega \frac{\tilde{u}^*}{r\sin\theta}\frac{\partial v\tilde{u}}{\partial\theta}\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi}(r\tilde{u}^* v\tilde{u}\sin\theta\partial r/\partial\theta)\,|_{r=r_s}\mathrm{d}\theta\mathrm{d}\lambda - \int_\Omega \tilde{u}vr^{-1}\partial\tilde{u}^*/\partial\theta\mathrm{d}\Omega, \tag{22}$$

$$\int_\Omega \frac{\tilde{u}^*}{r^2}\frac{\partial r^2 w\tilde{u}}{\partial r}\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} -(r\tilde{u}^* u\tilde{u}\partial r/\partial\lambda + r\tilde{u}^* v\tilde{u}\sin\theta\partial r/\partial\theta)\,|_{r=r_s}\mathrm{d}\theta\mathrm{d}\lambda - \int_\Omega \tilde{u}w\partial\tilde{u}^*/\partial r\mathrm{d}\Omega, \tag{23}$$

(21) + (22) + (23) yields

$$\int_\Omega(\prod\tilde{u})\tilde{u}^*\mathrm{d}\Omega = \int_\Omega\tilde{u}(-\Lambda\tilde{u}^*)\mathrm{d}\Omega. \tag{24}$$

In a similar way, one has

$$\int_\Omega(\Lambda\tilde{u})\tilde{u}^*\mathrm{d}\Omega = \int_\Omega\tilde{u}(-\prod\tilde{u}^*)\mathrm{d}\Omega. \tag{25}$$

Thus, one gets (17) by use of (24) + (25).

We now prove (18). Because

$$\int_\Omega(l'_1 G\tilde{T})\tilde{u}^*\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi}\int_{r_s}^{r_\infty}\frac{\tilde{u}^*}{\rho^* r\sin\theta}\frac{\partial G\tilde{T}}{\partial\lambda}r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\lambda$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_{r_s}^{r_\infty}\Big[\frac{\partial(r\tilde{u}^* G\tilde{T}\rho^{*-1})}{\partial\lambda} - \frac{G\tilde{T}}{r\sin\theta}\frac{\partial(\tilde{u}^*\rho^{*-1})}{\partial\lambda}r^2\sin\theta\Big]\mathrm{d}r\mathrm{d}\theta\mathrm{d}\lambda$$

$$= \int_0^{2\pi}\int_0^{\pi} r\tilde{u}^* G\tilde{T}\rho^{*-1}\partial r/\partial\lambda\,|_{r=r_s}\mathrm{d}\theta\mathrm{d}\lambda - \int_\Omega\tilde{T}Gl_1\tilde{u}^*\mathrm{d}\Omega, \tag{26}$$

$$\int_\Omega(l'_2 G\tilde{T})\tilde{v}^*\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} r\tilde{v}^* G\tilde{T}\sin\theta\rho^{*-1}\partial r/\partial\theta\,|_{r=r_s}d\theta d\lambda - \int_\Omega\tilde{T}Gl_2\tilde{v}^*\mathrm{d}\Omega, \tag{27}$$

$$\int_\Omega(l'_3 G\tilde{T})\tilde{w}^*\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} -(r\tilde{u}^* G\tilde{T}\rho^{*-1}\partial r/\partial\lambda$$

$$+ \tilde{rv}^* G\tilde{T}\sin\theta\rho^{*-1}\partial r/\partial\theta)\mid_{r=r_s}\mathrm{d}\theta\mathrm{d}\lambda - \int_\Omega \tilde{T}Gl_3\tilde{w}^*\mathrm{d}\Omega, \tag{28}$$

(26) + (27) + (28) yields (18). In analogy to that, we can obtain (19). Therefore,

$$N^*(\varphi) = \begin{bmatrix} -\mathscr{R} & -(f+u\mathrm{ctg}\theta/r) & -(\tilde{f}+u/r) & 0 & -l'_1 G \\ f+u\mathrm{ctg}\theta/r & -\mathscr{R} & -v/r & 0 & -l'_2 G \\ \tilde{f}+u/r & v/r & -\mathscr{R} & -g/\sqrt{2}\phi^* & -l'_3 G \\ 0 & 0 & g/\sqrt{2}\phi^* & -\mathscr{R} & 0 \\ -Gl_1 & -Gl_2 & -Gl_3 & 0 & -\mathscr{R} \end{bmatrix}, \tag{29}$$

and similarly one has $L(\varphi) = L^*(\varphi)$. The proof is complete.

**Lemma 3.** $L(\varphi)$ *is symmetrical and* $N(\varphi)$ *anti-symmetrical, i.e.*

$$(L(\varphi)\varphi_1,\varphi_2) = (\varphi_1, L(\varphi)\varphi_2),\ (N(\varphi)\varphi_1,\varphi_2) = -(\varphi_1, N(\varphi)\varphi_2),\ \forall\, \varphi,\varphi_1,\varphi_2 \in H_0(\Omega). \tag{30}$$

Since

$$(L(\varphi)\varphi,\varphi) = \int_\Omega \Big\{\hat{\mu}(\nabla\cdot V^*)^2 + (\mu+k_\lambda)\nabla u^*\cdot\nabla u^*$$

$$+ (\mu+k_\theta)\nabla v^*\cdot\nabla v^* + (\mu+k_r)\Delta w^*\cdot\nabla w^*$$

$$+ K_\lambda r^{-2}\sin^{-2}\theta(\partial T^*/\partial\lambda)^2 + K_\theta r^{-2}(\partial T^*/\partial\theta)^2 + K_r(\partial T^*/\partial r)^2\Big\}\mathrm{d}\Omega$$

$$+ \int_{S^2\times\{r_s\}} [C_{DK}(\mu+k_{r_s})V^*\cdot V^* + 2^{-1}(K_{r_s}+\kappa_{r_s}C_V^{-1})\alpha_S T^{*2}]\, r_S^2\sin\theta\mathrm{d}\theta\mathrm{d}\lambda, \tag{31}$$

where $V^* = v^*\theta^\circ + u^*\lambda^\circ + w^* r^\circ$, one has

**Lemma 4.**

$$(L(\varphi)\varphi,\varphi) \geqslant 0, \tag{32}$$

*where the equality is true if and only if* $\|\varphi\|_0 = 0$.

**Lemma 5.** *There is a constant* $C > 0$ *such that*

$$\|\varphi\|_0^2 \leqslant C\|\psi\|_0^2, \tag{33}$$

$\forall\ \varphi = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\rho}, \tilde{T})',\ \psi = (u^*, v^*, w^*, \rho^*, T^*)' \in H_0(\Omega)$.

**Lemma 6.** *There is a constant* $C_1 > 0$ *such that*

$$C_1 \mid \psi \mid_1^2 \leqslant (\mathrm{L}(\varphi)\varphi, \varphi), \tag{34}$$

$$C_1 \mid \psi \mid_0^2 \leqslant (\mathrm{L}(\varphi)\varphi, \varphi), \tag{35}$$

*where* $\varphi = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\rho}, \tilde{T})'$, $\varphi = (u^*, v^*, w^*, \rho^*, T^*)' \in H_0(\Omega)$, $\mid \psi \mid_1 = (\| u^* \|_{H^1}^2 + \| v^* \|_{H^1}^2 + \| w^* \|_{H^1}^2 + \| T^* \|_{H^1}^2)^{1/2}$, $\mid \varphi \mid_0 = (\| u^* \|^2 + \| v^* \|^2 + \| w^* \|^2 + \| T^* \|^2)^{1/2}$.

*Proof.* According to formula (31), we only have to prove

$$C\left(\int_{\partial\Omega_1} f^{*2}\mathrm{d}s + \mid f^* \mid_1^2\right) \geqslant \| f^* \|_0^2, \tag{36}$$

for certain constant $C > 0$, where $f = u, v, w$ or $T$.

Let $f^* = gq$, in which $q$ is an unknown function, and $g$ an undetermined function. Then

$$\begin{aligned}
\int_\Omega \nabla f^* \cdot \nabla f^* \mathrm{d}\Omega &= \int_\Omega \Big[ g^2 \nabla q \cdot \nabla q - q^2 g \Delta g + \frac{1}{r^2 \sin^2\theta} \frac{\partial}{\partial\lambda}\left(q^2 g \frac{\partial g}{\partial \lambda}\right) \\
&\quad + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(q^2 g \sin\theta \frac{\partial g}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(q^2 g r^2 \frac{\partial g}{\partial r}\right)\Big] \mathrm{d}\Omega \\
&\geqslant \int_\Omega \Big[ - q^2 g \Delta g + \frac{1}{r^2 \sin^2\theta} \frac{\partial}{\partial\lambda}\left(q^2 g \frac{\partial g}{\partial \lambda}\right) \\
&\quad + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(q^2 g \sin\theta \frac{\partial g}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(q^2 g r^2 \frac{\partial g}{\partial r}\right)\Big] \mathrm{d}\Omega \\
&\geqslant - \int_\Omega q^2 g \Delta g \mathrm{d}\Omega + \int_{\partial\Omega_2} q^2 g \partial g / \partial r \mathrm{d}S^2 - \int_{\partial\Omega_1} q^2 g \partial g / \partial r \mathrm{d}S^2,
\end{aligned}$$

where $\partial\Omega_1 = S^2 \times \{r_S\}$, $\partial\Omega_2 = S^2 \times \{r_\infty\}$. Therefore,

$$- \int_\Omega q^2 g \Delta g \mathrm{d}\Omega \leqslant \int_\Omega \nabla f^* \cdot \nabla f^* \mathrm{d}\Omega + \left| \int_{\partial\Omega_1} q^2 g \partial g / \partial r \mathrm{d}S^2 \right| + \left| \int_{\partial\Omega_2} q^2 g \partial g / \partial r \mathrm{d}S^2 \right|,$$

i.e.

$$- \int_\Omega f^{*2} \Delta g / g \mathrm{d}\Omega \leqslant \int_\Omega \nabla f^* \cdot \nabla f^* \mathrm{d}\Omega + \left| \int_{\partial\Omega_1} f^{*2} g^{-1} \partial g / \partial r \mathrm{d}S^2 \right|$$

$$+ \left| \int_{\partial\Omega_2} f^{*2} g^{-1} \partial g / \partial r \mathrm{d}S^2 \right|. \tag{37}$$

From eq. (37), if $g$ exists so that

$$-\Delta g/g \geqslant k_1 > 0, \ | g^{-1}\partial g/\partial r |_{\partial\Omega_1} \leqslant k_2, \ | g^{-1}\partial g/\partial r |_{\partial\Omega_2} = 0, \tag{38}$$

where $k_1$ and $k_2$ are positive constants, then

$$k_1 \int_\Omega f^{*2} \mathrm{d}\Omega \leqslant \int_\Omega \nabla f^* \cdot \nabla f^* \mathrm{d}\Omega + k_2 \int_{\partial\Omega_1} f^{*2} \mathrm{d}S^2. \tag{39}$$

Take

$$c = \max(1/k_1, k_2/k_1),$$

and we have inequality (36). For the cases of $f = u, v$ or $w$, taking

$$K_2 = c^{-1}\min(\mu + K_\lambda, \ \mu + K_\theta, \ \mu + K_r, \ C_{DK}(\mu + k_r)),$$

and for $f = T$, taking

$$\tilde{K}_2 = c^{-1}\min(K_\lambda, K_\theta, K_r, 2^{-1}(K_{r_S} + \kappa_{r_S} C_V^{-1})\alpha_S),$$

then we get inequality (35).

There are many functions which satisfy (38), for example, one can take

$$g = \cos\theta \sin\frac{\lambda}{2} \sin\frac{\pi a}{2r^*}, \tag{40}$$

where $a = r_\infty + b$, $r^* = r + b$ and $b > 0$ satisfying $b \neq (2n r_S + r_\infty)/2n$, and then gets

$$k_1 = \left( \frac{9}{4r_\infty^2} + \frac{\pi^2}{4(r_\infty + b)^2} \right),$$

where $k_2$ is a certain positive constant. The proof is complete.

## 3 Results

According to Lemmas 1—6, the following results are obtained. Since

$$\| \psi \|_0^2 \leqslant \| \psi \|_1^2,$$

$$C_1 | \psi |_1^2 + C_1 \| \rho^* \|^2 = C_1 \| \psi \|_1^2,$$

by Lemmas 5 and 6, one has

$$\frac{d}{dt}\|\varphi\|_0^2 + 2\tilde{C}\|\varphi\|_0^2 \leqslant 2(C_m + |\zeta(t)|), \tag{41}$$

where $\tilde{C}$ is a positive constant,

$$C_m = C_1\|\rho^*\|^2, \tag{42}$$

$$|\zeta| = 2^{-1}\int_\Omega (|\varepsilon| + |\hat{C}\alpha_S T_S|)d\Omega, \tag{43}$$

where $\hat{C}$ is given in (6). By integrating the continuity equation over $\Omega$ and using the known BVCs we get $\|\rho^*\|^2 = \int_\Omega \rho d\Omega$ = constant, which is another expression of the conservation law of mass. $C_m$ therefore is a constant. From (41) and by the classical Gronwall inequality, we have

**Theorem 1.** *The solution $\varphi$ of the operator equations* (1) *and* (2) *with the BVCs* (7)—(11) *satisfies*

$$\|\varphi(t)\|_0^2 \leqslant \left\{\|\varphi\|_0^2 + 2\int_0^t e^{\tilde{c}t}\left(C_m + |\zeta|\right)dt\right\}e^{-\tilde{c}t}, \ t \in [0, T], \tag{44}$$

*where $\tilde{C}$, $C_m$ and $|\zeta|$ are given in* (41), (42) *and* (43) *respectively.*

With Theorem 1 one can easily get the global absorbing set as follows.

**Theorem 2.** *Let*

$$B_k = \left\{\varphi = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\rho}, \tilde{T})' \in H_0(\Omega) \middle| \ \|\varphi\|_0^2 \leqslant K^2)\right\}, \tag{45}$$

*where $\tilde{M} = 2(C_m + |\zeta|)\tilde{C}^{-1}$, $K > \tilde{M}^{1/2}$. The solution $\varphi$ of eqs.* (1) *and* (2) *satisfies*

(i) *if $\varphi_0 \in B_k$, then $\varphi(t) \in B_k$ for $\forall t \geqslant 0$;*

(ii) *if $\varphi_0 \notin B_k$, then $\varphi(t) \in B_k$ for $\varphi(t) \in B_k$,*

*where $\tau = \tilde{C}^{-1}\ln(\|\varphi_0\|_0^2 - \tilde{M}K^{-1} - \tilde{M})$.*

The existence of the global absorbing set $B_k$ reveals that the atmospheric system is a dissipative system, and that the property of dissipative structure is a basic characteristic of atmospheric motion. Points out of the $B_k$ represent transient process, which shows that there is the irreversible characteristic in the atmospheric system. Using Theorem 2 we may further get Theorem 3.

**Theorem 3.** *There exists a global attractor A in the operator equations of* (1) *and* (2).

The theorem shows that there is still a global attractor $A$ in the full atmospheric equations while considering topographic effect. The existence of the global attractor indicates the nonlinear adjustment of the atmosphere to the surface temperature $T_S$ (e.g. 43), topographic effect ($\hat{C}$ in (43)) and external forcing ($\varepsilon$ in (43)).

## 4 Conclusion

Based on the full atmospheric equations, the asymptotic behavior of the solutions of the atmospheric equations with topographic effect is investigated. With the properties of integral with parameters, the properties of the operators of the atmospheric equations are studied in the basic function spaces, and then the existence theorem of global attractor of the atmospheric equations is obtained. Therefore, the known results of the qualitative theory of the atmospheric equations without topographic effect are extended to the case with topographic effect. It must be noted that the conclusions in this paper hold also for moist atmosphere.

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