Vol. 42 No. 4 SCIENCE IN CHINA (Series D) August 1999

Inertial manifold of the atmospheric equations*

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Received July 26, 1998

Abstract For a class of nonlinear evolution equations, their global attractors are studied and the existence of their inertial manifolds is discussed using the truncated method. Then, on the basis of the properties of operators of the atmospheric equations, it is proved that the operator equation of the atmospheric motion with dissipation and external forcing belongs to the class of nonlinear evolution equations. Therefore, it is known that there exists an inertial manifold of the atmospheric equations if the spectral gap condition for the dissipation operator is satisfied. These results furnish a basis for further studying the dynamical properties of global attractor of the atmospheric equations and for designing better numerical scheme.

Keywords: **inertial manifold, nonlinear evolution equation, global attractor, operator equation, operator.**

Results of qualitative theory on the nonlinear atmospheric dynamics[1—14] show that the atmosphere system, whether it is dry or moist, whether there is a topographic effect on it or not, and whether its external forcing is stationary or non-stationary, will evolve to a global attractor as time increases. The long-time behavior of solutions of the system depends on the global attractor, and points out of the attractor, which have nothing to do with the asymptotic state of the system while time tends to infinity, have only transient sense. The existence of the atmosphere attractor reveals that there is a nonlinear adjustment to external forcing in the system, and that the asymptotic behavior of solutions of the full atmospheric dynamical equations can be accurately described by a finite-dimensional ordinary differential equation, which furnishes necessary mathematical and physical foundations for setting and designing the models of long-term numerical weather forecast and numerical climate forecast. This attractor, however, may be a fractal and an unsmooth manifold. Moreover, its speed of attraction for orbits of solutions of the system is not exponential. These bring some difficulties for further dynamical analysis and practical computation. Therefore, it is important to find an invariant smooth manifold which contains the global attractor and attracts all the orbits of solutions of the system with exponentially speed. This is another important basic concept of inertial manifold for characterizing asymptotic properties of solutions apart from the concept of global attractor in the studies of infinite-dimensional dynamical system in recent years[15—17]. There are significant senses not only for analyzing dynamical properties on global attractor, but also for designing reasonable discretization numerical scheme in

* Project supported by the Project on Mechanism and Prediction Theory of Heavy Climate Disaster in China and the State Key Project on Dynamics and Prediction Theory of Climate.

investigations on inertial manifold. It is natural to ask whether there exists an inertial manifold in the atmospheric equations or not. The question is mainly discussed in this paper.

1 Definition of inertial manifold

Definition 1. Assume that the semigroup operator $\{S(t)\}_{t\geqslant 0}$ has a global attractor A. Then subset $M_I \subset H$ (H is a Hilbert space) is called an inertial manifold of A if it satisfies

(i) M_I is a finite-dimensional smooth manifold (at least a Lipshitz manifold);

(ii) M_I is invariant, i.e.

$$S(t)M_I \subset M_I; \tag{1}$$

(iii) M_I attracts all the orbits of solutions for semigroup $\{S(t)\}_{t\geqslant 0}$ with exponential speed, namely for $u_0 \in H$ there exist constants $k_1,\ k_2 > 0$ such that

$$\mathrm{dis}t(S(t)u_0, M_I) \leqslant k_1 \mathrm{e}^{-k_2 t},\ \forall t \geqslant 0; \tag{2}$$

(iv) the global attractor A of $S(t)$ is on M_I.

2 Attractors and manifolds of a class of nonlinear evolution equations

2.1 A class of nonlinear evolution equations

Given a Hilbert space H with inner product $(\cdot,\cdot)$ and norm $|\cdot|$, we study a class of nonlinear evolution equations as follows:

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} + L\vartheta + R(\vartheta) = 0, \tag{3}$$

where

$$R(\vartheta) = \delta_1 B(\vartheta,\vartheta) + \delta_2 B_1(\vartheta,\vartheta) + \delta_3 A\vartheta - f, \tag{4}$$

where $\delta_i\ (i = 1,2,3)$ satisfy

$$\delta_i = 0 \text{ or } 1,\ \text{and } \delta_1 + \delta_2 \geqslant 1. \tag{5}$$

The case for $\delta_2 = 0$ has been discussed[15—17], but the case for $\delta_2 = 1$ is first presented and investigated in this paper. The linear operator L is an unbounded positively definite self-adjoint operator in H, $D(L)$ is dense in H, and L^{-1} is compact. The mapping $\vartheta \to L\vartheta$ is an isomorphism from $D(L)$ onto H. Let L^s be s powers ($s \in \mathbb{R}$) of L, and then the spaces $V_{2s} = D(L^s)$ are Hilbert spaces with the following inner products:

$$(\vartheta_1,\vartheta_2)_{2s} = (L^s\vartheta_1, L^s\vartheta_2),\ \forall \vartheta_1,\vartheta_2 \in D(L^s), \tag{6}$$

$\vartheta \in V_s$. Let

$$|\vartheta|_s = (\vartheta,\vartheta)_s^{1/2}. \tag{7}$$

Since L^{-1} is a compact self-adjoint operator, it follows from the Rellich inference[15—17] that there exist an orthonormal basis $\{w_j\}_{j=1}^{\infty}$ and the eigenvalues λ_j of L in H such that

$$Lw_j = \lambda_j w_j,\ (j = 1,\ 2,\ \cdots) \tag{8}$$

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \lambda_{j+1} \leqslant \cdots, \tag{9}$$

$$\lim_{\lambda_j \to \infty} \lambda_j = +\infty. \tag{10}$$

From (8) and (9), it is easy to obtain

$$|L^{1/2}\vartheta| \geqslant \lambda_1^{1/2} |\vartheta|,\ \forall \vartheta \in D(L^{1/2}), \tag{11}$$

$$|L^{s+1/2}\vartheta| \geqslant \lambda_1^{1/2} |L^s\vartheta|,\ \forall \vartheta \in D(L^{s+1/2}),\ \forall s. \tag{12}$$

Letting $P = P_N$ be an orthogonal projection from H to Span$\{w_1, \cdots, w_N\}$, $Q_N = I - P_N$, one has

$$\lambda_1 |p| \leqslant |Lp| \leqslant \lambda_N |p|, \; p \in PD(L), \tag{13}$$

$$|Lq| \geqslant \lambda_{N+1} |q|, \; q \in QD(L). \tag{14}$$

$B(\vartheta, \vartheta)$ and $B_1(\vartheta, \vartheta)$ are bilinear operators of $D(L) \times D(L) \to H$, and A is a linear operator of $D(L) \to H$. Suppose that $B(\vartheta, \vartheta)$, $B_1(\vartheta, \vartheta)$ satisfy

$$(\delta_1 B(\vartheta, \vartheta_1) + \delta_2 B_1(\vartheta, \vartheta_1), \vartheta_1) = 0, \; \forall \vartheta, \vartheta_1 \in D(L), \tag{15}$$

$$|B(\vartheta, \vartheta_1)| \leqslant C_1 |\vartheta|^{1/2} |L^{1/2}\vartheta|^{1/2} |L^{1/2}\vartheta_1|^{1/2} |L\vartheta_1|^{1/2}, \; \forall \vartheta, \vartheta_1 \in D(L), \tag{16}$$

$$|B_1(\vartheta, \vartheta_1)| \leqslant C_2 |L^{1/2}\vartheta|^{1/2} |L^{1/2}\vartheta_1|^{1/2}, \; \forall \vartheta, \vartheta_1 \in D(L). \tag{17}$$

Then A satisfies one of the following properties:

$$|A\vartheta| \leqslant C_3 |L^{1/2}\vartheta|, \; \forall \vartheta \in D(L), \tag{18}$$

$$|A\vartheta| \leqslant C_4 |L^{1/2}\vartheta|^{1/2} |L\vartheta|^{1/2}, \; \forall \vartheta \in D(L). \tag{19}$$

Additionally, B, B_1 and A also have the continuity properties:

$$|L^{1/2}B(\vartheta, \vartheta_1)| \leqslant C_5 |L\vartheta| |L\vartheta_1|, \; \forall \vartheta, \vartheta_1 \in D(L), \tag{20}$$

$$|L^{1/2}B_1(\vartheta, \vartheta_1)| \leqslant C_6 |L\vartheta|^{1/2} |L\vartheta_1|^{1/2}, \; \forall \vartheta, \vartheta_1 \in D(L), \tag{21}$$

$$|L^{1/2}A\vartheta| \leqslant C_7 |L\vartheta|, \; \forall \vartheta \in D(L), \tag{22}$$

where C_i $(i = 1, \cdots, 7)$ are positive constants. Besides, assume that $A + L$ is positively definite, i.e. there exists a constant $C_8 > 0$ such that

$$((A + L)\vartheta, \vartheta) \geqslant C_8 |L^{1/2}\vartheta|^2, \; \forall \vartheta \in D(L). \tag{23}$$

If A is anti-adjoint, i.e.

$$(A\vartheta, \vartheta) = 0, \; \forall \vartheta \in D(L), \tag{24}$$

then it is obvious that (23) is true. However, A in (23) may not be an anti-adjoint operator.

2.2 Global attractor

We now consider the initial-value problem of (3), namely (3) has the following initial condition:

$$\vartheta(0) = \vartheta_0 \in H. \tag{25}$$

Assume that there exist a unique solution $S(t)\vartheta_0 \in D(L)$ $(\forall t \in \mathbb{R}^+)$ for the initial-value problems (3) and (23). The mapping $S(t)$ has properties of semigroup as usual.

Based on the properties of the operators mentioned above, one has the following lemma.

Lemma 1. *For any initial value $\vartheta_0 \in H$, there are ρ_0, ρ_1 and ρ_2 which depend on λ_1, $|f|$ and $|L^{1/2}f|$ such that*

$$\limsup_{t \to +\infty} |\vartheta(t)|^2 \leqslant \rho_0^2, \tag{26}$$

$$\limsup_{t \to +\infty} |L^{1/2}\vartheta(t)|^2 \leqslant \rho_1^2, \tag{27}$$

$$\limsup_{t \to +\infty} |L\vartheta(t)|^2 \leqslant \rho_2^2. \tag{28}$$

Lemma 1 implies that any solution of (3) will run respectively into the following balls after some time $t \geqslant \tau > 0$:

$$B_0 = \{\vartheta \in H, \; |\vartheta| \leqslant 2\rho_0\}; \tag{29}$$

$$B_1 = \{\vartheta \in D(L^{1/2}), \; |L^{1/2}\vartheta| \leqslant 2\rho_1\}; \tag{30}$$

$$B_2 = \{\vartheta \in D(L), \; |L\vartheta| \leqslant 2\rho_2\}. \tag{31}$$

Thus we have Theorem 1.

Theorem 1. *There exists a global attractor* $\mathscr{A}$ *in* (3). *It is a limiting set of* B_2, *i.e.*

$$\mathscr{A} = \omega(B_2) = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s} S(t)B_2}, \tag{32}$$

where the closure is taken in H, *and we have*

$$\mathscr{A} \subseteq B_2 \cap B_1 \cap B_0. \tag{33}$$

2.3 Inertial manifold

Here we discuss inertial manifolds of (3) by use of the truncated method of ref. [15]. Assume that $\theta(s)$ is a smooth function of $\mathbb{R}^+ \to [0,1]$:

$$\begin{cases} \theta(s) = \begin{cases} 1, & s \in [0,1]; \\ 0, & s \geqslant 2; \end{cases} \\ |\theta'(s)| \leqslant 2, \quad s \geqslant 0. \end{cases} \tag{34}$$

Fix $\rho = 2\rho_2$, and define

$$\theta_\rho(s) = \theta(s/\rho), \; s \geqslant 0. \tag{35}$$

Then the truncated equation of (3) is

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} + L\vartheta + F(\vartheta) = 0, \tag{36}$$

where $F(u) = \theta_\rho(|L\vartheta|)R(\vartheta)$. Clearly, when $|L\vartheta| \leqslant \rho$, $\theta_\rho(|L\vartheta|) = 1$, and in this case (36) is accordant with (3). When $|L\vartheta| \geqslant 2\rho, \theta_\rho(|L\vartheta|) = 0$, and in this case making the inner product for the two sides of (36) with $L^2\vartheta$, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|L\vartheta|^2 + \lambda_1|L\vartheta|^2 \leqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|L\vartheta|^2 + \lambda_1|L^{3/2}\vartheta|^2 \leqslant 0. \tag{37}$$

Therefore, orbit $\vartheta(t)$ is exponentially convergent to the ball whose radius $\rho_3 \geqslant 2\rho$ in $D(L)$.

Let $p(t) = P\vartheta, q(t) = Q\vartheta$. Then p and q in PH and QH satisfy

$$\frac{\mathrm{d}p}{\mathrm{d}t} + Lp + PF(\vartheta) = 0, \tag{38}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} + Lq + QF(\vartheta) = 0, \tag{39}$$

where $\vartheta = p + q$, and the inertial manifold $M_I = \mathrm{Graph}(\Phi)$, i.e. it is got by the graph of the Lipschitz mapping Φ: $PD(L) \to QD(L)$. The mapping Φ is obtained by the fixed point of the function space $H_{b,l}$. The definition of the function space $H_{b,l}$ is as follows.

Definition 2. $H_{b,l}$ is a function space of the Lipschitz mapping Φ: $PD(L) \to QD(L)$ satisfying

$$|L\Phi(p)| \leqslant b, \; b > 0 \text{ is undetermined}, p \in PD(L); \tag{40}$$

$$|L\Phi(p_1) - L\Phi(p_2)| \leqslant l\,|Lp_1 - Lp_2|, \; l > 0, \; p_1, p_2 \in PD(L); \tag{41}$$

$$\mathrm{supp}\Phi \subseteq \{p \in PD(L) \mid |Lp| \leqslant 4\rho\}. \tag{42}$$

Introduce

$$\|\Phi_1 - \Phi_2\| = \sup_{p\in D(l)} |L\Phi_1(p) - L\Phi_2(p)|. \tag{43}$$

Then $H_{b,l}$ is a complete metric space.

For $\Phi \in H_{b,l}$, we define an inertial mapping T of $PD(L)$:

$$T\Phi(p_0) = -\int_{-\infty}^{0} \mathrm{e}^{\tau LQ} QF(\vartheta)\mathrm{d}\tau, \; p_0 \in PD(L), \tag{44}$$

where $\vartheta(\tau) = p(\tau;\Phi,p_0) + \Phi(p(\tau;\Phi,p_0))$, $p(\tau;\Phi,p_0)$ is a solution of (36) satisfying $p(\tau;\Phi,p_0) = p_0$. Hence, for (3) one has Theorem 2[12].

Theorem 2. *Suppose that operators A, B, B_1 and L satisfy* (8)—(16), (20)—(23), *and* (18) (*or* (19)), $0 < l < 1/8$, *and that there exist constants N_0, K_1 and K_2 (which depend on l and intial values) satisfying $N \geqslant N_0$, $\lambda_{N+1} \geqslant K_1$ and the spectral gap condition $\lambda_{N+1} - \lambda_N \geqslant K_2$. Then there exists $b > 0$ such that*

(i) *T maps $H_{b,l}$ into $H_{b,l}$*;

(ii) *T has a fixed point in $H_{b,l}$*;

(iii) *M_l = Graph(Φ) is an inertial manifold of* (3);

(iv) *M_l contains the global attractor of* (3).

3 Inertial manifold of the atmospheric equations

3.1 Description of the equations

In coordinate (λ,θ,p), the atmospheric equations of large-scale motion are as follows[1—3]:

$$\frac{\partial u}{\partial t} + \Lambda u + \left(2\Omega\cos\theta + \frac{\operatorname{ctg}\theta}{a}u\right)v + \frac{1}{a\sin\theta}\frac{\partial\phi}{\partial\lambda} + L_1 u = 0, \tag{45}$$

$$\frac{\partial v}{\partial t} + \Lambda v - \left(2\Omega\cos\theta + \frac{\operatorname{ctg}\theta}{a}u\right)u + \frac{1}{a}\frac{\partial\phi}{\partial\theta} + L_1 v = 0, \tag{46}$$

$$\frac{\partial\phi}{\partial p} + \frac{R}{p}T = 0, \tag{47}$$

$$\frac{1}{a\sin\theta}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial v\sin\theta}{\partial\theta}\right) + \frac{\partial\omega}{\partial p} = 0, \tag{48}$$

$$\frac{R^2}{C^2}\frac{\partial T}{\partial t} + \frac{R^2}{C^2}\Lambda T - \frac{R}{p}\omega + L_2 T = \frac{R^2}{C^2}\frac{\varepsilon}{C_p}, \tag{49}$$

where

$$\Lambda = \frac{u}{a\sin\theta}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\theta} + \omega\frac{\partial}{\partial p},$$

$$L_i = -\frac{\partial}{\partial p}l_i\frac{\partial}{\partial p} - \mu_i\nabla^2 \quad (i = 1, 2),$$

$$l_i = \upsilon_i(gp/R\bar{T})^2 \quad (i = 1, 2),$$

$$\nabla^2 = \frac{1}{a\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{a^2\sin^2\theta}\frac{\partial^2}{\partial\lambda^2},$$

$$C^2 = R^2\bar{T}(\gamma_d - \gamma)/g,$$

$\bar{T} = \bar{T}(p)$ is the mean temperature over isobaric p surface, T the deviation from $\bar{T}$, ϕ the deviation from $\bar{\phi}$, ε the diabatic heating, and other symbols are meteorologically as usual. The domain of solutions of the equations is $\Omega = S^2 \times (p_0, P_s)$, $0 < p_0 < P_s < \infty$, where p_0 is a certain small positive number, and P_s the atmospheric pressure on the earth's surface. The boundary value conditions are

$$p = p_0, \ (\partial u/\partial p, \partial v/\partial p, \ \omega, \ \partial T/\partial p) = 0, \tag{50}$$

$$p = p_s, \ u = v = \omega = 0, \tag{51}$$

$$\partial T/\partial p = \alpha_s(T_s - T), \tag{52}$$

where T_s is the temperature on the earth's surface, α_s a parameter related to the turbulent thermal conductivity. We only discuss the case of the homogeneous boundary value condition of (52), i.e.

$$\partial T/\partial p = -\alpha_s T. \tag{53}$$

For the non-homogeneous case, it may be discussed by using homogeneous transformation. Here we do not consider the topographic effect, so $\phi(\lambda, \theta, P_s) = 0$.

The initial value condition is

$$(u, v, T)|_{t=0} = (u^{(0)}, v^{(0)}, T^{(0)}). \tag{54}$$

From (48) and the boundary value conditions, one has

$$\omega(\lambda, \theta, p) = -\int_p^{P_s}(\partial\omega/\partial p)\mathrm{d}p = -\int_p^{P_s}((\partial u/\partial\lambda + \partial v\sin\theta/\partial\theta)/a\sin\theta)\mathrm{d}p. \tag{55}$$

Similarly, one has

$$\phi(\lambda, \theta, p) = -\int_p^{P_s}(\partial\phi/\partial p)\mathrm{d}p = -\int_p^{P_s}(RT/p)\mathrm{d}p. \tag{56}$$

Equations (45)—(49) are essentially the equations of three variable u, v and T, i.e. (45), (46) and (49), where ω and ϕ are given by (55) and (56). Therefore, by introducing the vector function

$$\vartheta = (u, v, (R/C)T)^{\mathrm{T}}, \tag{57}$$

eqs. (45)—(49) can be written as the following operator equation:

$$\frac{\partial\vartheta}{\partial t} + R(\vartheta) + L\vartheta = 0, \tag{58}$$

where

$$R(\vartheta) = B(\vartheta)\vartheta + B_1(\vartheta)\vartheta + A\vartheta - \zeta, \tag{59}$$

$$L = \mathrm{diag}(L_1, L_1, C^2L_2/R^2), \tag{60}$$

$$B(\vartheta) = \begin{bmatrix} \Lambda_1 & u\,\mathrm{ctg}\theta/a & 0 \\ -u\,\mathrm{ctg}\theta/a & \Lambda_1 & 0 \\ 0 & 0 & \Lambda_1 \end{bmatrix}, \tag{61}$$

$$B_1(\vartheta) = \mathrm{diag}\left(\omega\frac{\partial}{\partial p}, \omega\frac{\partial}{\partial p}, \omega\frac{\partial}{\partial p}\right). \tag{62}$$

When A is a linear operator,

$$A\vartheta = \left(fv + \frac{1}{a\sin\theta}\frac{\partial\phi}{\partial\lambda}, -fu + \frac{1}{a}\frac{\partial\phi}{\partial\theta}, -\frac{C}{p}\omega\right)^{\mathrm{T}}, \tag{63}$$

$$(A\vartheta_1, \vartheta_2) = \int_\Omega\left[\left(fv_1 + \frac{1}{a\sin\theta}\frac{\partial\phi_1}{\partial\lambda}\right)u_2 + \left(-fu_1 + \frac{1}{a}\frac{\partial\phi_1}{\partial\theta}\right)v_2 + \left(-\frac{R}{p}\omega_1\right)T_2\right]\mathrm{d}\Omega, \tag{64}$$

where ω and ϕ are given by (55) and (56), i.e.

$$\omega_i(\lambda, \theta, p) = -\int_p^{P_s}((\partial u_i/\partial\lambda + \partial v_i\sin\theta/\partial\theta)/a\sin\theta)\mathrm{d}p, \quad (i = 1,2),$$

$$\phi_i(\lambda, \theta, p) = -\int_p^{P_s}(RT_i/p)\mathrm{d}p, \quad (i = 1,2),$$

$$\Lambda_1 = \frac{u}{a\sin\theta}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\theta}, \tag{65}$$

$$\xi = (0, 0, R\varepsilon/(CC_p))^{\mathrm{T}}. \tag{66}$$

The diabatic heating ε is given and stationary in this paper, and at this same time there exists a unique solution for the initial-boundary value problems (45)—(54)[5]. A semigroup $S(t)$ is therefore defined by the operator equation (58) for the initial value $\vartheta(0) = \vartheta_0$.

3.2 Properties of operators

Let H be a Hilbert space (taking $H = L_2(\Omega)$ in the following parts), $D(L)$ the domain of definition of operator L corresponding to the boundary value conditions (50), (51) and (53) (being marked as $D_{\partial\Omega}$). Then $D(L) = \{\vartheta \mid \vartheta \in C^\infty(\bar{\Omega}),\ D_{\partial\Omega}\}$, where $D(L)$ is a linear dense set.

Lemma 2. *Operator L is symmetrical, namely,*

$$(L\vartheta_1, \vartheta_2) = (\vartheta_1, L\vartheta_2), \quad \forall \vartheta_1, \vartheta_2 \in D(L). \tag{67}$$

Lemma 3. *Operator L is self-adjoint.*

As is easily seen, one has

$$(L\vartheta, \vartheta) \geqslant 0, \tag{68}$$

for $\vartheta \in D(L)$. Furthermore, there exist constants C_1, $C_2 > 0$ such that

$$(L\vartheta, \vartheta) \geqslant C_1 \| \vartheta \|_1, \tag{69}$$

$$(L\vartheta, \vartheta) \geqslant C_2 \| \vartheta \|_0, \tag{70}$$

where

$$\| \vartheta \|_1 = (\| u \|_{H^1}^2 + \| v \|_{H^1}^2 + \| T \|_{H^1}^2)^{1/2}, \tag{71}$$

$$\| \vartheta \|_0 = (\| u \|^2 + \| v \|^2 + \| T \|^2)^{1/2}, \tag{72}$$

where $\| \cdot \|_{H^1}$ takes the $H^1(\Omega)$-norm, $\| \cdot \|$ the $L_2(\Omega)$-norm. Therefore, we have Lemma 4.

Lemma 4. *Operator L is positively definite, i.e. there exists $C > 0$ such that*

$$(L\vartheta, \vartheta) \geqslant C(\vartheta, \vartheta), \quad \forall \vartheta \in D(L), \tag{73}$$

where the equality is true if and only if $\vartheta = 0$.

Based on the above analyses, L is a self-adjoint positively definite linear operator. Thus one can introduce a new inner product in $D(L)$ as follows:

$$(\vartheta_1, \vartheta_2) = (L\vartheta_1, \vartheta_2). \tag{74}$$

This yields a new norm

$$\| \vartheta \|_1 = (\vartheta, \vartheta)_1^{1/2} = (L\vartheta, \vartheta)^{1/2}, \quad \vartheta \in D(L). \tag{75}$$

A new complete Hilbert space can be got by taking the closure of $D(L)$ according to (75). Obviously, $D(L)$ is dense in H_1. Using the positively definite property of L, one has

$$\| \vartheta \|_0 \leqslant C^{-1} \| \vartheta \|_1, \ \vartheta \in D(L). \tag{76}$$

H_1 is therefore embedded in H.

More generally, let L^s be s powers ($s \in \mathbb{R}$) of L, and the spaces $H_{2s} = D(L^s)$ be the Hilbert spaces with the following inner product:

$$(\vartheta_1, \vartheta_2)_{2s} = (L^s\vartheta_1, L^s\vartheta_2), \quad \forall \vartheta_1, \vartheta_2 \in D(L^s), \tag{77}$$

$\vartheta \in H_s$. Besides,

$$\| \vartheta \|_s = (\vartheta, \vartheta)_s^{1/2}. \tag{78}$$

Lemma 5. *The inverse operator L^{-1} of operator L is compact.*

Proof. For $\forall f \in H$, there exists a unique solution $\vartheta = L^{-1}f$ of $L\vartheta = f$ in $D(L)$. And using

$$\| \vartheta \|^2 \leqslant C(L\vartheta, \vartheta) = C(f, \vartheta),$$

one has

$$\| \vartheta \| \leqslant C^* \| f \|. \tag{79}$$

L^{-1} is therefore a bounded linear operator of $H \to H_1$. In addition, the embedding from H_1 to H is compact, so L^{-1} is compact.

L^{-1} is a self-adjoint compact operator because both L and L^{-1} are self-adjoint. It follows that there exists an orthonormal basis $\{W_j\}_{j=1}^{\infty}$ and the eigenvalues $\hat{\lambda}_j > 0$ of L and $\hat{\lambda}_j > \hat{\lambda}_{j+1}$ such that

$$L^{-1} W_j = \hat{\lambda}_j W_j \quad (j = 1,2,\cdots). \tag{80}$$

Let $\lambda_k = \hat{\lambda}_j^{-1}$. Then one has

$$L W_j = \lambda_j W_j,\ W_j \in D(L),\ (j = 1,2,\cdots), \tag{81}$$

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \lambda_{j+1} \leqslant \cdots, \tag{82}$$

$$\lim_{j \to \infty} \lambda_j = \infty. \tag{83}$$

Therefore, L has properties the same as those of (11)—(14).

For operators B, B_1 and A we have Lemma 6.

Lemma 6.

$$(B(\vartheta, \vartheta_1) + B_1(\vartheta, \vartheta_1), \vartheta_1) = 0, \tag{84}$$

$$(A\vartheta, \vartheta) = 0. \tag{85}$$

Let

$$B(\vartheta, \vartheta_1) = B(\vartheta)\vartheta_1, \tag{86}$$

$$B_1(\vartheta, \vartheta_1) = B_1(\vartheta)\vartheta_1. \tag{87}$$

Then both $B(\vartheta, \vartheta_1)$ and $B_1(\vartheta, \vartheta_1)$ are linear operators of $D(L) \times D(L) \to H$. Thanks to the Holder inequality and the Sobolev embedded inequality, we have Lemma 7.

Lemma 7. *There exists a constant* $C_1 > 0$ *such that*

$$|B(\vartheta, \vartheta_1)| \leqslant C_1 \|\vartheta\|_0^{1/2} \|\vartheta\|_1^{1/2} \|\vartheta_1\|_1^{1/2} \|\vartheta_1\|_2^{1/2}, \tag{88}$$

where

$$\|\vartheta_1\|_2 = (\|u\|_2^2 + \|v\|_2^2 + \|T\|_2^2)^{1/2}. \tag{89}$$

According to the Friedrichs inequality, one has

$$\|\omega\| \leqslant K_1 \|\partial\omega/\partial p\|, \tag{90}$$

where constant $K_1 > 0$. Moreover, using the Minkowski inequality, one has

$$\|\omega\| \leqslant K_1(|u|_1 + |v|_1). \tag{91}$$

Similarly, one can get

$$\|\phi\| \leqslant K_2 \|\partial\phi/\partial p\|, \tag{92}$$

$$\|\phi\| \leqslant K_2 \|RT/C\|, \tag{93}$$

where constant $K_2 > 0$. Additionally, thanks to the Schwarz inequality, we have Lemma 8.

Lemma 8. *There exist constants* k, $K > 0$ *such that*

$$|B_1(\vartheta, \vartheta_1)| \leqslant k \|\vartheta\|_1 \|\vartheta_1\|_1, \tag{94}$$

$$|A\vartheta| \leqslant K \|\vartheta\|_1. \tag{95}$$

3.3 Inertial manifold

It is easy to see that Lemma 1 is true for eq. (58) because operators in (58) satisfy the properties of operators in (3) according to lemmas 2—8. Furthermore, Theorem 2 is true for eq. (58) if the spectral gap condition is satisfied. This shows that there exist an global attractor and an inertial manifold M_I in the atmospheric equations (45)—(49) under that condition, and the inertial manifold contains the attractor and attracts all the orbits of solutions of eqs. (45)—(49) with exponential speed.

4 Conclusion

Global attractors and inertial manifolds of a class of nonlinear evolution equations are studied in this paper. And it is proved that the atmospheric equations just belong to this kind of nonlinear evolution equations, and that there exists an inertial manifold of the atmospheric equations if the spectral gap condition for the dissipation operator is satisfied. These results furnish a basis for understanding further the dynamical construction and characteristic of the global attractor of the atmosphere because the inertial manifold is an invariant smooth manifold which attracts all the orbits of solutions with exponential speed. Additionally, we may use the existence of inertial manifold to design a better numerical scheme to simulate precisely the long-term variation of the atmosphere. If there exists an inertial manifold $M_{\tau I}$ (here τ is the time stepsize) in the discretization numerical scheme designed and $M_{\tau I} \to M_I$ as $\tau \to 0$, this kind of numerical scheme is able to simulate precisely the long-term behavior of solutions of the original equations. This will be reported in another paper.

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