Operator constraint principle for simplifying atmospheric dynamical equations

LI Jianping[1, 2] & CHOU Jifan[2]

1. National Key Laboratory of Atmospheric Sciences and Geophysical Fluid Dynamics (LASG), Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing 100029, China;
2. Department of Atmospheric Sciences, Lanzhou University, Lanzhou 730000, China

Correspondence should be addressed to Li Jianping (email: ljp@lasg.iap.ac.cn)

Abstract **Based on the qualitative theory of atmospheric dynamical equations, a new method for simplifying equations, the operator constraint principle, is presented. The general rule of the method and its mathematical strictness are discussed. Moreover, the way that how to use the method to simplify equations rationally and how to get the simplified equations with harmonious and consistent dynamics is given.**

Keywords: simplification, operator equation, operator constraint principle.

The atmospheric dynamical equations are very complicated nonlinear, non-stationary and compressible partial differential equations with dissipation and external forcing[1—4]. They describe the various spatial-temporal motions in the atmosphere. At present, many difficulties cannot be overcome to solve the equations analytically under the proper initial-boundary value conditions. However, the special spatial-temporal motion in the atmosphere has its own way, so some factors could be neglected when we simplify equations for particular atmospheric motion. This is not only easy for us to make mathematical analysis, but we can also lay stress on the essential of motion and grasp the heart of problem.

The scale analysis developed since the 1940s has been a main and broadly adoptive method with semi-experimental property for simplifying equations[1—7]. The method assumes that each term in the same equation does not possess equal importance, thereby equations may be simplified according to the rule that the minor terms are left out by comparing the order of magnitude of every term in the same equation. An apparent deficiency of the scale analysis is that the method only compares the order of magnitude of each term in the same equation and disregards the links between equations in the original equations. Consequently, we probably get a set of inconsistent simplified equations without consistent properties in the original equations. Applying the method to simplifying equations, sometimes we can find that the same term is regarded as a secondary term and omitted in a certain equation, but it is retained in another one. It is difficult to explain this result in physical sense, therefore, results obtained by the semi-experimental theoretical analysis method are usually coarse and loose[3—7]. In order to hold the original properties of the system, other constraint methods need to be used. The energy constraint[8] is simply an effective method with clear physical sense. The method, however, is only available for the adiabatic system without friction.

To make up for the lack of the scale analysis and get consistently simplified dynamical equations, an operator constraint method is proposed in this study. This method is an extension of the energy constraint. The method is based on the fact that the atmospheric system is an essentially dissipative structure and is constructed on the qualitative theory of atmospheric dynamical equations[9−19]1), and abides by the rule that the properties of corresponding operators in the original and simplified equations should be kept unchanged, thereby the simplified equations obtained will not distort the essential properties of original equations.

1 General principle of the operator constraint method

The full atmospheric dynamical equations can be rewritten as an equivalent operator equation in Hilbert space[17, 18] 1):

$$\begin{cases}\dfrac{\partial\varphi}{\partial t}+N(\varphi)\varphi+L(\varphi)\varphi=\xi\\ \varphi|_{t=0}=\varphi_0\end{cases}, \tag{1}$$

where the vector function $\varphi=(\tilde u,\tilde v,\tilde w,\tilde\rho,\tilde T)'$, the symbol $'$ denotes transposition, $(\tilde u,\tilde v,\tilde w,\tilde\rho,\tilde T)'=\rho^*(u^*,v^*,w^*,\phi^*,T^*)'$, $u^*=u/\sqrt{2}$, $v^*=v/\sqrt{2}$, $w^*=w/\sqrt{2}$, $\phi^*=\sqrt{\phi}$, $\rho^*=\sqrt{\rho}$, $T^*=\sqrt{C_V\hat T}=\sqrt{C_V T}$, where u, v and w are zonal, meridional and vertical winds respectively, ρ and T denote density of air and air temperature respectively, C_v represents specific heat (for the concrete form of the operators $N(\varphi)$ and $L(\varphi)$ see refs. [17, 18] and footnote 1)). The operator $N(\varphi)$ represents the nonlinear advection, Coriolis force, pressure-gradient force, gravity, spherical curvature, etc. $L(\varphi)$ embodies the effects of dissipative terms. From the most universal sense, the abstract operator $N(\varphi)$ is an anti-adjoint and antisymmetrical operator, the $L(\varphi)$ is a self-adjoint and symmetrical operator, namely

$$N(\varphi)=-N^*(\varphi)\,,\quad (N(\varphi)\varphi_1,\varphi_2)=-(\varphi_1,N(\varphi)\varphi_2),$$
$$(N(\varphi_1)\varphi,\varphi)=0\,;$$
$$L(\varphi)=L^*(\varphi)\,,\quad (L(\varphi)\varphi_1,\varphi_2)=(\varphi_1,L(\varphi)\varphi_2),$$

1) Li Jianping, Qualitative theory of the dynamical equations of atmospheric and oceanic motion and its applications, Ph. D. Dissertation (in Chinese), Lanzhou University, 1997, 209.

$$(L(\varphi)\varphi,\varphi) \geqslant 0.$$

$\forall \varphi, \varphi_1, \varphi_2 \in H_0(\Omega)$, where $H_0(\Omega)$ is a complete Hilbert space with the following inner product and norm

$$(\varphi_1,\varphi_2) = \int_\Omega \varphi_1' \varphi_2 \mathrm{d}\Omega$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_{r_s}^{r_\infty} \varphi_1'\varphi_2 r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\lambda,$$

$$\|\varphi\|_0 = (\varphi\ ,\varphi)^{1/2}.$$

The physical senses implied by the properties of $N(\varphi)$ and $L(\varphi)$ as mentioned above are that $N(\varphi)$ represents the various kinds of reversible processes of energy conservation and $L(\varphi)$ shows the irreversible processes of energy dissipation. For a simplified equation, its corresponding operators should hold the properties of $N(\varphi)$ and $L(\varphi)$ so that the essential physical properties of the original equations will not be broken down. This is simply the operator constraint principle for simplifying equations. Let the simplified equation be

$$\frac{\partial\varphi}{\partial t} + \tilde{N}(\varphi)\varphi + \tilde{L}(\varphi)\varphi = \xi\ . \tag{2}$$

According to the rule mentioned above, the simplified operator $\tilde{N}(\varphi)$ should be an anti-adjoint operator and $\tilde{L}(\varphi)$ should be self-adjoint. We can prove that there is the same asymptotic behavior in eqs. (1) and (2). This shows that there is no false source or sink and there is still a global attractor in the simplified equation obtained by the principle. The simplified equation is still a "dissipative structure", which does not destroy the asymptotic behavior of the solution of original equations and holds the whole properties and physical laws of original equations. Hence, the simplified equations obtained by the operator constraint principle are still very strict and their dynamical relations are harmonious and consistent. For example, the large-scale atmospheric equations can be written as follows[9—11, 18]:

$$\frac{\partial B\varphi}{\partial t} + N_l(\varphi)\varphi + L_l\varphi = \xi\ . \tag{3}$$

The operators $N_l(\varphi)$ and L_l in eq. (3) do not lose the properties of $N(\varphi)$ and $L(\varphi)$[9—11, 18], so there is the same long-time behavior between eqs. (3) and (1) and the simplification is reasonable.

2 Simplification of the operator *N*

According to the discussions stated above, $N(\varphi)$ is an anti-symmetric operator, so is the simplified operator $\tilde{N}(\varphi)$.

Let n_{ij} be a term in $N(\varphi)$ and n_{ji} the anti-symmetric term of n_{ij}. To the problem considered and properties of the objective studied, if n_{ij} is regarded as minor and can be neglected, n_{ji} should also be omitted in order to hold the anti-symmetric property of $N(\varphi)$. Otherwise, there must be false "sources" or "sinks", thereby elementary properties of the original system are lost. Considering the system without dissipations and forcings, i.e. the adiabatic non-friction system, eq. (1) becomes

$$\frac{\partial\varphi}{\partial t} + N(\varphi)\varphi = 0\ . \tag{4}$$

Making H_0 inner production with φ, the energy conservation follows immediately, namely

$$\|\varphi(t)\| = \|\varphi_0\|\ . \tag{5}$$

For any simplified equation of (4),

$$\frac{\partial\tilde{\varphi}}{\partial t} + \tilde{N}(\tilde{\varphi})\tilde{\varphi} = 0\ , \tag{6}$$

if $\tilde{N}(\tilde{\varphi})$ still holds the anti-adjoint properties of $N(\varphi)$, it results immediately by making H_0 inner production with φ that there is also the energy conservation in the simplified equation (6), i.e.

$$\|\tilde{\varphi}(t)\| = \|\tilde{\varphi}_0\|\ . \tag{7}$$

This indicates that eq. (6) does not break down the energy conservation in the original equation. If $\tilde{N}$ does not possess the anti-adjoint property of N, it can be proved that the simplification destroys the energy conservation property of eq. (4). There are many examples such as the barotropic non-divergent model, the linear barotropic model, the shallow water model, and the *p*-coordinate primitive equation model, which satisfy the principle as indicated earlier. Additionally, two improper examples from the scale analysis are also given because they do not have the above properties.

It can be concluded from the above analysis that the energy constraint method is a special case of the operator constraint method and in the process of simplification the operator constraint method is more concise and much clearer than the energy method.

3 Simplification of the operator *L*

The operator $L(\varphi)$ is an asymmetric and positive operator, so is the simplified operator $\tilde{L}(\varphi)$.

Let l_{ij} be a certain term in $L(\varphi)$ and l_{ji} the asymmetric term of l_{ij}. If l_{ij} is left out, l_{ji} must be omitted. Otherwise, the asymmetric and non-negative property of $L(\varphi)$ will be lost, thereby physical properties of the original system are distorted. In the following two simplifications used are discussed. The boundary conditions may be either with or without topography[17, 18]1).

1) See the footnote on page 1053.

Because the turbulent viscosity force is much greater than the molecule viscosity force, the molecule viscosity in the motion equations is usually neglected while dissipation is taken account of. After the molecule viscosity is omitted, the operator $L(\varphi)$ can be rewritten as follows:

$$\tilde{L}(\varphi)=\begin{bmatrix} -l & 0 & 0 & 0 & 0 \\ 0 & -l & 0 & 0 & 0 \\ 0 & 0 & -l & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -l_5 \end{bmatrix}, \tag{8}$$

where l and l_5 are:

$$l=\frac{1}{\rho^* r\sin\theta}\frac{\partial}{\partial\lambda}\frac{k_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda}\frac{1}{\rho^*}+\frac{1}{\rho^* r\sin\theta}\frac{\partial}{\partial\theta}\frac{k_\theta\sin\theta}{r}\frac{\partial}{\partial\theta}\frac{1}{\rho^*}+\frac{1}{\rho^* r^2}\frac{\partial}{\partial r}k_r r^2\frac{\partial}{\partial r}\frac{1}{\rho^*},$$

$$l_5=\frac{1}{\rho^* r\sin\theta}\frac{\partial}{\partial\lambda}\frac{K_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda}\frac{1}{\rho^*}+\frac{1}{\rho^* r\sin\theta}\frac{\partial}{\partial\theta}\frac{K_\theta\sin\theta}{r}\frac{\partial}{\partial\theta}\frac{1}{\rho^*}+\frac{1}{\rho^* r^2}\frac{\partial}{\partial r}K_r r^2\frac{\partial}{\partial r}\frac{1}{\rho^*}-C_K\alpha_S T_S\big/2\tilde{T}^2,$$

respectively; k_λ, k_θ and k_r represent turbulent viscosity coefficients in horizontal and vertical directions respectively; K_λ, K_θ, K_r are turbulent conductivity in horizontal and vertical directions, respectively. It is easy to prove that there are the same properties between $\tilde{L}(\varphi)$ and $L(\varphi)$. Therefore, the simplification is appropriate.

After the molecule viscosity is left out, it is common to think that the turbulent viscosity in horizontal direction is much less than that in vertical direction. As a result, the turbulent viscosity in horizontal direction is also omitted. Is this simplification true? The analysis shows that after neglecting the turbulent viscosity in horizontal direction, $L(\varphi)$ becomes

$$\tilde{L}(\varphi)=\begin{bmatrix} -\tilde{l} & 0 & 0 & 0 & 0 \\ 0 & -\tilde{l} & 0 & 0 & 0 \\ 0 & 0 & -\tilde{l} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tilde{l}_5 \end{bmatrix}, \tag{9}$$

where

$$\tilde{l}=\frac{1}{\rho^* r^2}\frac{\partial}{\partial r}k_r\frac{\partial}{\partial r}\frac{1}{\rho^*}, \tag{10}$$

$$\tilde{l}_5=\frac{1}{\rho^* r^2}\frac{\partial}{\partial r}K_r\frac{\partial}{\partial r}\frac{1}{\rho^*}-C_K\alpha_s T_s\big/2\tilde{T}^2. \tag{11}$$

It may be proved that

$$\left(\tilde{L}(\varphi)\varphi_1,\varphi_2\right)=\left(\tilde{L}(\varphi)\varphi_2,\varphi_1\right), \tag{12}$$

that is to say, $\tilde{L}(\varphi)$ is asymmetric. Then, we need to prove whether

$$\left(\tilde{L}(\varphi)\varphi,\varphi\right)\geqslant 0. \tag{14}$$

The answer is positive. However, to prove formula (14) we need to introduce a new Hilbert $\tilde{H}_1(\Omega)$, which is a complete space with the following norm

$$\|\varphi\|_1=\left(\left|\tilde{u}\right|_1^2+\left|\tilde{v}\right|_1^2+\left|\tilde{w}\right|_1^2+\left|\tilde{\rho}\right|_0^2+\left|\tilde{T}\right|_1^2\right)^{1/2}, \tag{15}$$

where $|\bullet|_1$ takes the norm in the Hilbert space $V(\Omega)$. $V(\Omega)$ is a complete space with the following norm

$$|f|_1=\left(\int_\Omega\left[f^2+\left(\frac{\partial f}{\partial r}\right)^2\mathrm{d}\Omega\right]\right)^{1/2}. \tag{16}$$

Using the new space, we can prove formula (14). This indicates that simplification can only alter the state space but not the properties of operators, and consequently, it is clear that the simplified equation does not distort the evolution law of the original equation in the qualitative sense.

4 Summary

In the theoretical study of atmospheric dynamics the primitive equations often need to be simplified. Simplification changes state space, but properties of simplified operators should hold those of original operators. The rule can assure that simplification does not distort elementary physical laws and global properties of the original system. This is just the operator constraint principle presented by this note for simplifying atmospheric dynamics equations. According to our discussions, this method has a strict mathematical basis and definite physical sense. Simplified equations with harmonious and consistent dynamics can be obtained by applying the combined method of the scale analysis and the operator constraint principle. It should be noticed that, however, if we study the phenomenon related to long-term process of the atmosphere, according to the operator principle the simplified system should still be a forced dissipative nonlinear system, which should neither be an adiabatic system without friction nor a linear system[11—19]. Only in this way do the simplified equations not distort the long-time behavior of the original system and also can better results be obtained.

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